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On the Conserved Caginalp Phase-Field System with Logarithmic Potentials Based on the Maxwell–Cattaneo Law with Two Temperatures

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Abstract

Our aim in this article is to study generalizations of the conserved Caginalp phase-field system based on the Maxwell-Cattaneo law with two temperatures for heat conduction and with logarithmic nonlinear terms. We obtain well-posedness results and study the asymptotic behavior of the associated system. In particular, we prove the existence of the global attractor and prove the strict separation to the pure phases in two space dimensions. Furthermore, we give some numerical simulations, obtained with the FreeFem++ software [23], comparing the conserved Caginalp phase-field type model with regular and with logarithmic nonlinear terms.

Keywords Caginalp system · Maxwell–Cattaneo law · Two temperatures · Logarithmic nonlinear terms · Well-posedness · Dissipativity · Global attractor · Simulations

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1 Introduction

In this article, we are interested in the study of a conserved Caginalp phase-field model with logarithmic potentials based on Maxwell-Cattaneo law for the heat conduction with two temperatures.

Caginalp introduced in [6] (see also [7]) the following phase-field system

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta T, \tag{1.1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{1.2}$$

where u is the order parameter and T is the relative temperature (defined as $T = \tilde{T} - T_E$, where $\tilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature). These equations model phase transition processes such as melting solidification processes and have been studied, e.g., in [4,5,20] and [30]; see also, e.g., [2,14,15,18,43] and [45] for a similar phase-field model with a memory term. Equations (1.1)–(1.2) consist of the coupling of the Cahn–Hilliard equation introduced in [8] and [9] with the heat equation.

These equations are known as the conserved phase-field mode, in the sense that, when endowed with Neumann boundary conditions, the spatial average of the order parameter and of the temperature are conserved quantities.

In this paper, we consider the conserved Caginalp phase-field model proposed in [41], in which we consider the theory of two-temperature-generalized thermoelasticity proposed in [48] and based on the Maxwell–Cattaneo law.

The generalized heat equation (1.2) is based on the usual Fourier law for heat conduction. Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction, see [13]). To overcome this drawback, or at least to account for more realistic features, several alternatives to the Fourier law, based, for example, on the Maxwell–Cattaneo law or recent laws from thermomechanics, have been proposed and studied in, e.g., [24,25,35–38] and [39]. Indeed, introducing the enthalpy

$$H = u + T, \tag{1.3}$$

we can rewrite this equation as

$$\frac{\partial H}{\partial t} = \operatorname{div} q, \tag{1.4}$$

where q is the thermal flux vector, and, assuming the Fourier law

$$q = -\nabla T, \tag{1.5}$$

we recover (1.2).

In the late 1960's, several authors proposed a heat conduction theory based on two temperatures (see [10,11] and [12]). More precisely, one now considers the conductive temperature T and the thermodynamic temperature θ. For time-independent problems the difference between these temperatures is proportional to the heat supply; they thus coincide when there is no heat supply. However, for time-dependent problems, they are generally different even in the absence of heat supply: this is in particular the case for non-simple materials. In that case, the two temperatures are related as follows:

$$\theta = T - \Delta T \tag{1.6}$$

and (1.1) should be replaced by

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta(T - \Delta T). \tag{1.7}$$

In this article, we consider the theory of two-temperature-generalized thermoelasticity proposed in [48] and based on the Maxwell-Cattaneo law.

In that case, in order to obtain the corresponding generalized heat equation, one writes

$$\frac{\partial H}{\partial t} = -\mathrm{div}\, q, \tag{1.8}$$

and

$$H = u + \theta = u + T - \Delta T, \tag{1.9}$$

where the heat flux q satisfies the Maxwell-Cattaneo law [48],

$$q + \tau \frac{\partial q}{\partial t} = -\nabla T, \tau > 0. \tag{1.10}$$

In particular, it follows from (1.8) that

$$\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} = -\mathrm{div} \left(q + \tau \frac{\partial q}{\partial t} \right),$$

hence, in view of (1.10),

$$\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} = \Delta T. \tag{1.11}$$

In this paper, we reformulate the problem in terms of the order parameter u and the enthalpy H (see also [5] for the original conserved Caginalp phase-field system and [34] for the nonconserved model based on the Maxwell–Cattaneo law). In particular, introducing the enthalpy $H = u + \theta = u + T - \Delta T$, we can rewrite (1.2) and (1.7) in the form

We thus deduce from (1.9) and (1.11) the generalized heat equation

$$(I - \Delta) \left(\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) - \Delta T = -\tau \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}. \tag{1.12}$$

Here, the presence of the second derivative $\frac{\partial^2 u}{\partial t^2}$ makes the mathematical analysis of the equation particularly difficult and, to overcome such a difficulty, we will rewrite the equation in a different way, keeping the enthalpy H as unknown. Indeed, it follows from (1.9) and (1.11) that

$$(I-\Delta)\left(\tau\frac{\partial^2 H}{\partial t^2}+\frac{\partial H}{\partial t}\right)=\Delta(T-\Delta T),$$

hence

$$(I-\Delta)\left(\tau\frac{\partial^2 H}{\partial t^2}+\frac{\partial H}{\partial t}\right)-\Delta H=-\Delta u \tag{1.13}$$

and

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta u-\Delta f(u)=-\Delta H. \tag{1.14}$$

In [41], the authors studied the well-posedness of the conserved Caginalp system (1.13)–(1.14), for regular nonlinear terms f and Dirichlet boundary conditions. It is however important to note that, in phase transition, regular nonlinear terms actually are approximations of thermodynamically relevant logarithmic ones of the form

$$f(s)=-2\kappa_0 s+\kappa_1\ln\left(\frac{1+s}{1-s}\right), \tag{1.15}$$

with $s\in(-1,1)$ and $0<\kappa_1<\kappa_0$, which follow from a mean-field model (see [9,32]; in particular, the logarithmic terms correspond to the entropy of mixing).

To our knowledge, there is no result on the original conserved Caginalp phase-field system with the aforementioned logarithmic nonlinear terms (see however [20] for dynamic boundary conditions; in that case, the situation is very different from that of Neumann boundary conditions), whereas, [30] treats the original conserved Caginalp model with regular nonlinear terms; see also [22] for a more general system, with a nonlinear coupling between u and T.

The conserved Caginalp phase-field model was studied in [31] for the type III thermomechanics theory and in [33] for the Maxwell Cattaneo law (see also [41], with two temperatures), for regular nonlinear terms. Also, recently, the authors in [28] study the nonconserved Caginalp phase-field model based on Maxwell Cattaneo law with two temperatures and logarithmic potentials; note that, in that case, the strict separation from the pure phases (see below) is easier, as one can use the comparison principle for second-order parabolic equations for the equation for the order parameter.

In order to compare the logarithmic potentials with the cubic ones in the numerical simulations that we perform, we choose the following cubic polynomial nonlinear term $f(s)=s^3-.8s$, whose corresponding double-well potential $F(s)=(s^2-.8)^2/4$ has two minima $-.8$ and $.8$ (see Fig. 1, left), and the logarithmic nonlinear term $f(s)=-2\kappa_0 s+\kappa_1\ln(\frac{1+s}{1-s})$, with $(\kappa_0,\kappa_1)=(\ln(3),0.8)$, whose corresponding double-well potential $F(s)=\kappa_0(1-s^2)+\kappa_1(1-s)\ln(1-s)+(1+s)\ln(1+s)$ has two minima $-.8$ and $.8$ (see Fig. 1,right).

In this article, we consider the conserved phase-field model (1.13)–(1.14), with the logarithmic nonlinear terms (1.15). We first prove the existence of weak solutions to equations (1.13)–(1.14). To do so, we approximate the singular nonlinear terms by regular ones and prove the convergence of the solutions to the approximated problems to that to the limit singular one. Then, we prove the uniqueness of the solution, which allows us to define the corresponding semigroup and prove the existence of a global attractor. We then prove some higher-order regularity results which lead to a strict separation property in two space dimensions. This strict separation property is very important. On the one hand, it says that we actually have the same problem, but now with a regular nonlinear term (even better, a bounded and regular one). On the other hand, it says that, in the phase transition process, there is always some given amount of the other phase. Note that such a strict separation property is not known in three space dimensions, already for the sole Cahn–Hilliard equation. To prove it, we adapt the techniques introduced in [21]. The difference here is that we have two additional terms due to the presence of the Enthalpy H in which the regularity of H plays an important role in order to have the strict separation property. Finally, we write the spatial and time discretizations of (1.13)–(1.14), which allows us to compare (numerically) the conserved Caginalp type model with regular and with logarithmic nonlinear terms.

Notation

We denote by $((\,\cdot\,,\,\cdot\,))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We further set $((\,\cdot\,,\,\cdot\,))_{-1} = (((-\Delta)^{-\frac{1}{2}}\cdot, (-\Delta)^{-\frac{1}{2}}\cdot))$, with associated norm $\|\cdot\|_{-1}$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Dirichlet boundary conditions. Note that $\|\cdot\|_{-1}$ is equivalent to the usual H_{-1}-norm on $H^{-1}(\Omega) = H_0^1(\Omega)'$. More generally, $\|\cdot\|_X$ denotes the norm in the Banach space X.

Throughout the article, the same letter c, c' (and, sometimes, C) denotes (generally positive) constants which may vary from line to line. Similarly, the same letter Q denotes (positive) monotone increasing (with respect to each argument) functions which may vary from line to line.

Setting the problem We consider the following initial and boundary value problem, in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$ or 3, with boundary Γ:

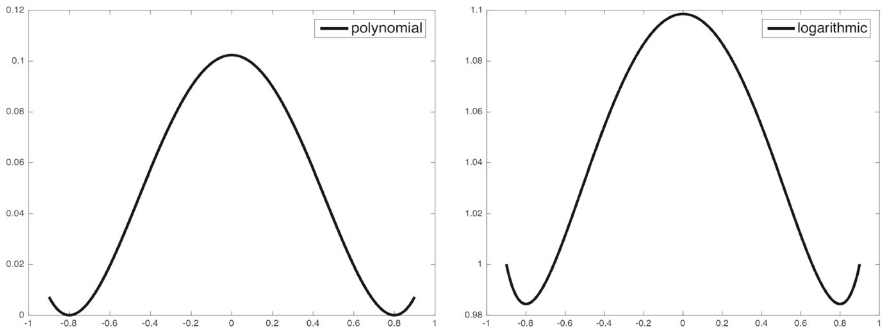

Fig. 1 Double-well potential, polynomial (left) and logarithmic (right)

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \varepsilon\Delta u + u + \frac{1}{\varepsilon}f(u) = H, \tag{1.16}$$

$$(I-\Delta)\left(\tau\frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t}\right) - \Delta H = -\Delta u, \tag{1.17}$$

$$u = H = 0 \quad \text{on} \quad \Gamma, \tag{1.18}$$

$$u|_{t=0} = u_0,\ H|_{t=0} = H_0,\ \frac{\partial H}{\partial t}|_{t=0} = H_1, \tag{1.19}$$

where, for simplicity, we have set ε and τ equals to one.

The potential f is defined by:

$$f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\frac{1+s}{1-s}\right), \tag{1.20}$$

with $s \in]-1, 1[$ and $0 < \kappa_1 < \kappa_0$. We then have

$$f'(s) = \frac{2\kappa_1}{1-s^2} - 2\kappa_0. \tag{1.21}$$

Furthermore, we see that

$$f \in \mathcal{C}^2(-1,1),\ f(0) = 0, \tag{1.22}$$

$$-c_0 \leqslant F(s) \leqslant f(s)s + c_0, \tag{1.23}$$

$$f(s)s \geqslant c_1|f(s)| - c_1, \tag{1.24}$$

where, $F(s) = \int_0^s f(\tau)\,d\tau$ and $c_0 \geqslant 0$,

$$f'(s) \geqslant -c_2,\ c_2 \geqslant 0. \tag{1.25}$$

Now, we will consider the following space

$$K = \{\varphi \in L^2(\Omega);\ -1 \leqslant \varphi \leqslant 1 \ \text{ a.e. in } \ \Omega\}.$$

Following the idea of Debussche and Dettori (see [16] and [17]), we consider the following approximated function $f_N \in \mathcal{C}^1(\mathbb{R})$, for $N \in \mathbb{N}$, by

$$f_N(s) = \begin{cases} f\left(-1+\frac{1}{N}\right) + f'\left(-1+\frac{1}{N}\right)\left(s+1-\frac{1}{N}\right), & s < -1+\frac{1}{N}, \\ f(s), & |s| \leqslant 1-\frac{1}{N}, \\ f\left(1-\frac{1}{N}\right) + f'\left(1-\frac{1}{N}\right)\left(s-1+\frac{1}{N}\right), & s > 1-\frac{1}{N}. \end{cases}$$

Then, we consider the approximated problem:

$$(-\Delta)^{-1}\frac{\partial u_N}{\partial t} - \Delta u_N + u_N + f_N(u_N) = H_N, \tag{1.26}$$

$$(I-\Delta)\left(\frac{\partial^2 H_N}{\partial t^2} + \frac{\partial H_N}{\partial t}\right) - \Delta H_N = -\Delta u_N, \tag{1.27}$$

$$u_N = H_N = 0 \quad \text{on} \quad \Gamma, \tag{1.28}$$

$$u_N|_{t=0} = u_0,\ H_N|_{t=0} = H_0,\ \frac{\partial H_N}{\partial t}|_{t=0} = H_1, \tag{1.29}$$

Recalling that, owing to [41], we have the following result concerning to the problem (1.26)–(1.29):

Theorem 1.1 *We assume that f_N satisfies (1.22)–(1.25) for $s \in \mathbb{R}$. Then, for every $(u_0, H_0, H_1) \in (H_0^1(\Omega))^3$, the problem (1.26)–(1.29) possesses a unique solution $(u_N, H_N, \frac{\partial H_N}{\partial t})$ such that*

$$\left(u_N, H_N, \frac{\partial H_N}{\partial t}\right) \in L^\infty(\mathbb{R}^+; H_0^1(\Omega))^3$$

and

$$\frac{\partial u_N}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \quad \forall T > 0.$$

We also note that, F is bounded and for $\varphi \in K$, we have $F_N(\varphi) \leqslant F(\varphi)$. Thus, all the estimations in this section are uniform with respect to N.

Remark 1.1 We can also endow the problem with periodic or Neumann boundary conditions. In these cases, we have, integrating (1.14) over Ω,

$$\frac{d\langle u\rangle}{dt} = 0, \tag{1.30}$$

where $\langle\cdot\rangle$ denotes the spatial average, hence

$$\langle u(t)\rangle = \langle u_0\rangle, \quad \forall t \geqslant 0, \tag{1.31}$$

Similarly, integrating (1.17) over Ω, we obtain

$$\frac{d}{dt}\left(\frac{d\langle H\rangle}{dt} + \langle H\rangle\right) = 0, \tag{1.32}$$

which yields

$$\frac{d\langle H\rangle}{dt} + \langle H\rangle = \langle H_0 + H_1\rangle \tag{1.33}$$

and

$$\langle H(t)\rangle = \langle H_0 + H_1\rangle - \langle H_1\rangle e^{-t}, \quad t \geqslant 0. \tag{1.34}$$

Taking (1.38)–(1.34) into account, we can adapt the proofs below and derive the same well-posedness results. Note however that, in order to study the existence of attractors, we need to assume that

$$|\langle u_0\rangle| \leqslant M_1, \tag{1.35}$$

$$|\langle H_0 + H_1\rangle| \leqslant M_2, \quad |\langle H_1\rangle| \leqslant M_3. \tag{1.36}$$

It thus follows from (1.31) and (1.34) that

$$|\langle u(t)\rangle| \leqslant M_1, \quad \forall t \geqslant 0, \tag{1.37}$$

$$|\left(\frac{d\langle H\rangle}{dt} + \langle H\rangle\right)(t)| \leqslant M_2, \quad |\frac{d\langle H\rangle}{dt}(t)| \leqslant M_3, \quad |\langle H(t)\rangle| \leqslant M_2 + M_3, \quad \forall t \geqslant 0. \tag{1.38}$$

We can then define the family of solving operators

$$S(t) : \Phi_M \to \Phi_M, \quad (u_0, H_0, H_1) \mapsto (u(t), H(t), \frac{\partial H}{\partial t}(t)), \quad t \geq 0,$$

where

$$\Phi_M(= \Phi_{M_1,M_2,M_3}) = \Big\{(\varphi, \theta, \xi) \in H^2(\Omega)^3, \ \frac{\partial \varphi}{\partial \nu} = 0 \ \text{ on } \ \Gamma; \\ \|\varphi\|_{L^\infty(\Omega)} < 1, \ |\langle\varphi\rangle| \leqslant M_1, \ |\langle\theta + \xi\rangle| \leqslant M_2, \ |\langle\xi\rangle| \leqslant M_3\Big\}.$$

We refer the interested reader to [40] for more details on the necessary modifications.

2 A Priori Estimates

We start by assuming *a priori* that:

$$\|u_0\|_{L^\infty(\Omega)} \leqslant 1 - \delta, \quad \delta \in (0, 1), \tag{2.1}$$

where δ is a fixed positive constant.

We multiply (1.26) by $\frac{\partial u_N}{\partial t}$, (1.27) by $(-\Delta)^{-1}\frac{\partial H_N}{\partial t}$ and summing the resulting equalities, to obtain

$$\frac{d}{dt}\left(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx + \|u_N - H_N\|^2 + \left\|\frac{\partial H_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial H_N}{\partial t}\right\|^2\right) \\ + 2\left(\left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial H_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial H_N}{\partial t}\right\|^2\right) = 0. \tag{2.2}$$

Next, we multiply (1.26) by u_N, (1.27) by $(-\Delta)^{-1}H_N$ and have, summing the resulting inequalities, owing to (1.24),

$$\frac{d}{dt}\Big(\|u_N\|_{-1}^2+\|H_N\|_{-1}^2+\|H_N\|^2+2\Big(\Big(\frac{\partial H_N}{\partial t},H_N\Big)\Big)_{-1}$$
$$+2\Big(\Big(\frac{\partial H_N}{\partial t},H_N\Big)\Big)\Big)+c\Big(\|u_N-H_N\|^2$$
$$+\|\nabla u_N\|^2+2\|f_N(u_N)\|_{L^1(\Omega)}\Big)\leqslant 2\Big(\Big\|\frac{\partial H_N}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial H_N}{\partial t}\Big\|^2\Big)+c',\ c>0. \tag{2.3}$$

Summing finally (2.2) and δ_1 times (2.3), where $\delta_1>0$ is chosen small enough, we have a differential inequality of the form

$$\frac{d}{dt}E_1^N+c\Big(E_1^N+\|f_N(u_N)\|_{L^1(\Omega)}+\Big\|\frac{\partial u_N}{\partial t}\Big\|_{-1}^2\Big)\leqslant c',\ c>0, \tag{2.4}$$

where

$$E_1^N=\|\nabla u_N\|^2+2\int_\Omega F_N(u_N)\,dx+\|u_N-H_N\|^2+\Big\|\frac{\partial H_N}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial H_N}{\partial t}\Big\|^2$$
$$+\delta_1\Big(\|u_N\|_{-1}^2+\|H_N\|_{-1}^2+\|H_N\|^2+2\Big(\Big(\frac{\partial H_N}{\partial t},H_N\Big)\Big)_{-1}+2\Big(\Big(\frac{\partial H_N}{\partial t},H_N\Big)\Big)\Big), \tag{2.5}$$

satisfies

$$E_1^N\geqslant c\Big(\|u_N\|_{H^1(\Omega)}^2+\int_\Omega F_N(u_N)\,dx+\|H_N\|^2+\Big\|\frac{\partial H_N}{\partial t}\Big\|^2\Big)-c',\ c>0. \tag{2.6}$$

We now multiply (1.27) by $\frac{\partial H_N}{\partial t}$, to obtain

$$\frac{d}{dt}\Big(\|\nabla H_N\|^2+\Big\|\frac{\partial H_N}{\partial t}\Big\|_{H^1(\Omega)}^2\Big)+\Big\|\frac{\partial H_N}{\partial t}\Big\|_{H^1(\Omega)}^2\leqslant\|\nabla u_N\|^2. \tag{2.7}$$

Multiplying also (1.27) by H_N, we find

$$\frac{d}{dt}\Big(\|H_N\|_{H^1(\Omega)}^2+2\Big(\Big(\frac{\partial H_N}{\partial t},H_N\Big)\Big)+2\Big(\Big(\nabla\frac{\partial H_N}{\partial t},\nabla H_N\Big)\Big)\Big)+\|\nabla H_N\|^2$$
$$\leqslant\|\nabla u_N\|^2+2\Big\|\frac{\partial H_N}{\partial t}\Big\|_{H^1(\Omega)}^2. \tag{2.8}$$

Summing (2.4), δ_2 times (2.7) and δ_3 times (2.8), where $\delta_2,\ \delta_3 > 0$ are chosen small enough, we have a differential inequality of the form

$$\frac{d}{dt}E_2^N + c\left(E_2^N + \|f_N(u_N)\|_{L^1(\Omega)} + \left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2\right) \leqslant c',\ c > 0, \tag{2.9}$$

where

$$\begin{aligned} E_2^N &= E_1^N + \delta_2\left(\|\nabla H_N\|^2 + \left\|\frac{\partial H_N}{\partial t}\right\|_{H^1(\Omega)}^2\right) \\ &\quad + \delta_3\left(\|H_N\|_{H^1(\Omega)}^2 + 2\left(\left(\frac{\partial H_N}{\partial t}, H_N\right)\right) + 2\left(\left(\nabla\frac{\partial H_N}{\partial t}, \nabla H_N\right)\right)\right), \end{aligned} \tag{2.10}$$

satisfies

$$E_2^N \geqslant c\left(\|u_N\|_{H^1(\Omega)}^2 + \int_\Omega F_N(u_N)\,dx + \|H_N\|_{H^1(\Omega)}^2 + \left\|\frac{\partial H_N}{\partial t}\right\|_{H^1(\Omega)}^2\right) - c',\ c > 0. \tag{2.11}$$

We note that (2.9) and Gronwall's lemma imply the dissipative estimate

$$E_2^N(t) \leqslant e^{-ct}E_2^N(0) + c',\ c > 0,\ t \geqslant 0. \tag{2.12}$$

Consequently, for N large enough,

$$\begin{aligned} &\|u_N(t)\|_{H^1(\Omega)}^2 + \|H_N\|_{H^1(\Omega)}^2 + \left\|\frac{\partial H_N}{\partial t}\right\|_{H^1(\Omega)}^2 \\ &\quad \leqslant e^{-ct}\left(\|u_0\|_{H^1(\Omega)}^2 + \|H_0\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2\right) + c'. \end{aligned} \tag{2.13}$$

Integrating now (2.9) with respect to time, we have, for $r > 0$ fixed,

$$\begin{aligned} &\|f_N(u_N)\|_{L^1((t,t+r)\times\Omega)} \\ &\quad \leqslant ce^{-c't}\left(\|u_0\|_{H^1(\Omega)}^2 + \|H_0\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2 + \int_\Omega F_N(u_0)\,dx\right) + c''(r). \end{aligned} \tag{2.14}$$

Furtheremore, for every $r > 0$,

$$\begin{aligned} &\int_t^{t+r}\left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 d\tau \\ &\quad \leqslant ce^{-c't}\left(\|u_0\|_{H^1(\Omega)}^2 + \|H_0\|_{H^1(\Omega)}^2 + \|H_1\|_{H^1(\Omega)}^2 + \int_\Omega F_N(u_0)\,dx\right) \\ &\quad + c''(r),\ c' > 0,\ t \geqslant 0. \end{aligned} \tag{2.15}$$

Also note that $\int_\Omega F_N(u_0)\,dx \leqslant c$, since, $\|u_0\|_{L^\infty(\Omega)} \leqslant 1-\delta$. Therefore, (2.14) yields have

$$\|f_N(u_N)\|_{L^1((t,t+r)\times\Omega)} \leqslant ce^{-c't}\left(\|u_0\|^2_{H^1(\Omega)} + \|H_0\|^2_{H^1(\Omega)} + \|H_1\|^2_{H^1(\Omega)} + c''(r)\right). \tag{2.16}$$

We have thus found an estimate on the L^1-norm of $f_N(u_N)$.

We finally multiply (1.26) by $-\Delta u_N$ and ontain, owing to (1.25) and classical elliptic regularity results,

$$\frac{d}{dt}\|u_N\|^2 + c\|u_N\|^2_{H^2(\Omega)} \leqslant c'(\|\nabla u_N\|^2 + \|H_N\|^2),\ c > 0. \tag{2.17}$$

In a second step, we differentiate (1.26) with respect to time to have the initial and boundary value problem

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u_N}{\partial t} - \Delta\frac{\partial u_N}{\partial t} + \frac{\partial u_N}{\partial t} + f'_N(u_N)\frac{\partial u_N}{\partial t} = \frac{\partial H_N}{\partial t}, \tag{2.18}$$

$$\frac{\partial u_N}{\partial t} = 0 \quad \text{on} \quad \Gamma, \tag{2.19}$$

$$\frac{\partial u_N}{\partial t}(0) = \Delta u_0 - u_0 - f(u_0) + H_0. \tag{2.20}$$

Multiplying (2.18) by $\frac{\partial u_N}{\partial t}$, we obtain, in view of (1.25),

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_N}{\partial t}\right\|^2_{-1} + c\left\|\frac{\partial u_N}{\partial t}\right\|^2_{H^1(\Omega)} \leqslant c_0\left\|\frac{\partial u_N}{\partial t}\right\|^2 + \left(\left(\frac{\partial H_N}{\partial t}, \frac{\partial u_N}{\partial t}\right)\right), \tag{2.21}$$

which yields, employing the interpolation inequality

$$\left\|\frac{\partial u_N}{\partial t}\right\|^2 \leqslant c\left\|\frac{\partial u_N}{\partial t}\right\|_{-1}\left\|\frac{\partial u_N}{\partial t}\right\|_{H^1(\Omega)}, \tag{2.22}$$

the differential inequality

$$\frac{d}{dt}\left\|\frac{\partial u_N}{\partial t}\right\|^2_{-1} + c\left\|\frac{\partial u_N}{\partial t}\right\|^2_{H^1(\Omega)} \leqslant c'\left(\left\|\frac{\partial H_N}{\partial t}\right\|^2 + \left\|\frac{\partial u_N}{\partial t}\right\|^2_{-1}\right). \tag{2.23}$$

This yields an $L^\infty(L^2)$-regularity on $(-\Delta)^{-1}\frac{\partial u_N}{\partial t}(0) \in L^2(\Omega)$, which, in view of (2.20), essentially means that $u_0 \in H^3(\Omega) \cap H^1_0(\Omega)$. This is not satisfactory, in particular, in view of the study of the dissipativity and the existence of (finite dimensional) attractors.

We multiply now (1.27) by $-\Delta\frac{\partial H_N}{\partial t}$ and $-\Delta H_N$ to obtain

$$\frac{d}{dt}\left(\|\Delta H_N\|^2+\left\|\nabla\frac{\partial H_N}{\partial t}\right\|^2+\left\|\Delta\frac{\partial H_N}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial H_N}{\partial t}\right\|^2+\left\|\Delta\frac{\partial H_N}{\partial t}\right\|^2\leqslant\|\Delta u_N\|^2 \tag{2.24}$$

and

$$\frac{d}{dt}\left(\|\nabla H_N\|^2+\|\Delta H_N\|^2+2\left(\left(\nabla\frac{\partial H_N}{\partial t},\nabla H_N\right)\right)+2\left(\left(\Delta\frac{\partial H_N}{\partial t},\Delta H_N\right)\right)\right)+\|\Delta H_N\|^2$$
$$\leqslant\|\Delta u_N\|^2+2\left(\left\|\nabla\frac{\partial H_N}{\partial t}\right\|^2+\left\|\Delta\frac{\partial H_N}{\partial t}\right\|^2\right), \tag{2.25}$$

respectively. Summing (2.24) and δ_4 times (2.25), where $\delta_4>0$ is chosen small enough, we find a differential inequality of the form

$$\frac{dE_3^N}{dt}+cE_3^N\leqslant c'\|\Delta u_N\|^2, \tag{2.26}$$

where

$$E_4^N=\|\Delta H_N\|^2+\left\|\nabla\frac{\partial H_N}{\partial t}\right\|^2+\left\|\Delta\frac{\partial H_N}{\partial t}\right\|^2$$
$$+\delta_4\left(\|\nabla H_N\|^2+\|\Delta H_N\|^2+s\left(\left(\nabla\frac{\partial H_N}{\partial t},\nabla H_N\right)\right)+2\left(\left(\Delta\frac{\partial H_N}{\partial t},\Delta H_N\right)\right)\right) \tag{2.27}$$

satisfies

$$E_3^N\geqslant c\left(\|H_N\|^2_{H^2(\Omega)}+\left\|\frac{\partial H_N}{\partial t}\right\|^2_{H^2(\Omega)}\right),\ c>0. \tag{2.28}$$

Gronwall's lemma then yields that $H_N,\frac{\partial H_N}{\partial t}\in L^\infty(0,T;H^2(\Omega))$.

3 Existence of Solutions in the Case $n\leqslant 3$

Theorem 3.1 *Let $(u_0,H_0,H_1)\in(H^2(\Omega)\cap H_0^1(\Omega))^3$, then the problem (1.16)–(1.19) admits atleast one solution $(u,H,\frac{\partial H}{\partial t})$ such that, $\forall T>0$*

$$u\in L^\infty(\mathbb{R}^+;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)),$$
$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(\Omega))$$

and

$$\left(H,\frac{\partial H}{\partial t}\right)\in L^\infty(\mathbb{R}^+;(H^2(\Omega)\cap H_0^1(\Omega)))^2.$$

Furthermore, $\forall T > 0$, $\|u(t)\|_{L^\infty(\Omega)} \leqslant 1$, *the set* $\{x \in \Omega,\ |u(t,x)| \geqslant 1\}$ *is of null measure.*

Proof We have, owing to (2.13), that

$$\sup_{t\in[0,T]} \left\{ \|u_N(t)\|^2_{H^1(\Omega)} + \|H_N(t)\|^2_{H^1(\Omega)} + \left\| \frac{\partial H_N}{\partial t}(t) \right\|^2_{H^1(\Omega)} \right\} \leqslant c', \tag{3.1}$$

where c' is independent of N.

Letting N tends to $+\infty$ and considering a subsequence, we have by estimation (3.1)

$$u_N \to u \quad \text{weak star in} \quad L^\infty(0,T;H^1(\Omega)), \tag{3.2}$$

$$H_N \to H \quad \text{weak star in} \quad L^\infty(0,T;H^1(\Omega)), \tag{3.3}$$

$$\frac{\partial H_N}{\partial t} \to \frac{\partial H}{\partial t} \quad \text{weak star in} \quad L^\infty(0,T;H^1(\Omega)). \tag{3.4}$$

Integrating (2.9) between 0 and t, we have, in view of (3.1),

$$E_2^N(t) + c\int_0^t \left\| \frac{\partial u_N}{\partial t} \right\|^2_{-1} ds \leqslant c', \quad \forall t \in [0,T], \ c \geqslant 0, \tag{3.5}$$

where c' is independent of N. It thus follows that

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)). \tag{3.6}$$

Integrating (2.17) between 0 and t, we deduce

$$\|u_N(t)\|^2 + c\int_0^t \|u_N(t)\|^2_{H^2(\Omega)} ds \leqslant c, \quad \forall t \in [0,T], \tag{3.7}$$

where c is independent of N. This results that

$$\Delta u_N \rightharpoonup \Delta u \quad \text{weakly in } L^2((0,T)\times\Omega). \tag{3.8}$$

The only difficulty, when passing to the limit, is to pass to the limit in the nonlinear terms containing f_N. First, it follows from (2.16) that $f_N(u_N)$ is bounded, independently of N, in $L^1((0,T)\times\Omega)$. Then, it follows from the explicit expression of f_N that

$$\text{meas}(F_{N,M}) \leqslant \mu\left(\frac{1}{N}\right), \quad N \leqslant M,$$

where

$$F_{N,M} = \left\{ (t,x) \in (0,T)\times\Omega;\ |u_M(t,x)| > 1 - \frac{1}{N} \right\}$$

and

$$\mu(s) = \frac{c}{\min(|f(1-s)|, |f(s-1)|)},$$

where, here, the constant c is independent of N and M. Note that there holds

$$\int_0^T \int_\Omega |f_M(u_M)|\,dx\,dt \geqslant \int_{F_{N,M}} |f_M(u_M)|\,dx\,dt \geqslant c' \text{ meas } (F_{N,M}) \frac{1}{\mu(\frac{1}{N})}, \tag{3.9}$$

where the constant c' is independent of N and M.

Passing now to the limit M tends to $+\infty$ (employing Fatou's lemma on (3.9)) and the N tends to $+\infty$ (noting that $\lim\limits_{s\to 0} \mu(s) = 0$) to find

$$\text{meas}\{(t,x) \in (0,T) \times \Omega;\ |u(t,x)| \geqslant 1\} = 0,$$

so that

$$-1 < u(t,x) < 1 \quad \text{a.e. } (t,x). \tag{3.10}$$

Next, it follows from the above almost everywhere convergence of u_N, (3.10) and the explicit expression of f_N that

$$f_N(u_N) \to f(u) \text{ a.e. } \ (t,x) \in (0,T) \times \Omega \tag{3.11}$$

Multiplying now (1.26) by $f_N(u_N)$ and integrate over Ω, using the monotony of f_N, we obtain

$$\|f_N(u_N)\|^2 \leqslant c\left(\left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 + \|H_N\|^2 + \|u_N\|^2_{H^1(\Omega)}\right), \tag{3.12}$$

integrating (3.12) between 0 and t by (3.2), (3.3) and (3.6), we have

$$\|f_N(u_N)\|^2_{L^2((0,T)\times\Omega)} \leqslant c, \tag{3.13}$$

where c is independent of N. Thus, it follows from (3.11) that

$$f_N(u_N(t)) \to f(u) \quad \text{weakly in} \quad L^2((0,T) \times \Omega)), \tag{3.14}$$

which finishes the proof of the passage to the limit (the weak continuity property follows from Strauss's lemma, see, e.g., [47]). □

The following result gives us the uniqueness of the solution of the problem (1.16)–(1.19).

Theorem 3.2 *Under the assumptions of Theorem 3.1, the problem (1.16)–(1.19) admits a unique solution with the above regularity.*

Proof Let $\left(u^{(1)}, H^{(1)}, \frac{\partial H^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, H^{(2)}, \frac{\partial H^{(2)}}{\partial t}\right)$ be two solutions to (1.16)–(1.19) with initial data $\left(u_0^{(1)}, H_0^{(1)}, H_1^{(1)}\right)$ and $\left(u_0^{(2)}, H_0^{(2)}, H_1^{(2)}\right)$, respectively. We set

$$\left(u, H, \frac{\partial H}{\partial t}\right) = \left(u^{(1)}, H^{(1)}, \frac{\partial H^{(1)}}{\partial t}\right) - \left(u^{(2)}, H^{(2)}, \frac{\partial H^{(2)}}{\partial t}\right)$$

and

$$\left(u_0, H_0, H_1\right) = \left(u_0^{(1)}, H_0^{(1)}, H_1^{(1)}\right) - \left(u_0^{(2)}, H_0^{(2)}, H_1^{(2)}\right)$$

and have

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + u + f\left(u^{(1)}\right) - f\left(u^{(2)}\right) = H, \tag{3.15}$$

$$(I-\Delta)\left(\frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t}\right) - \Delta H = -\Delta u, \tag{3.16}$$

$$u = H = 0 \ \text{ on } \ \Gamma, \tag{3.17}$$

$$u|_{t=0} = u_0, \ H|_{t=0} = H_0, \ \frac{\partial H}{\partial t}|_{t=0} = H_1. \tag{3.18}$$

Multiplying (3.15) by u, (3.16) by $(-\Delta)^{-1}\frac{\partial H}{\partial t}$, summing the resulting inequalities, we obtain, in view of (1.25), a differential inequality of the form

$$\frac{dE}{dt} + \|u\|^2_{H^1(\Omega)} \leqslant c(\|u\|^2 + \|H\|^2), \tag{3.19}$$

where

$$E = \|u\|^2_{-1} + \|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2, \tag{3.20}$$

satisfies

$$E \geqslant c\left(\|u\|^2_{-1} + \|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2\right), \ c > 0. \tag{3.21}$$

Using finally the interpolation inequality

$$\|u\|^2 \leqslant c\|u\|_{-1}\|u\|_{H^1(\Omega)},$$

we find the differential inequality

$$\frac{dE}{dt} \leqslant cE, \tag{3.22}$$

hence, owing to (3.21)–(3.22) and Gronwall's lemma

$$\|u(t)\|^2_{-1} + \|H(t)\|^2 + \left\|\frac{\partial H}{\partial t}(t)\right\|^2 \leqslant ce^{c't}(\|u_0\|^2_{-1} + \|H_0\|^2 + \|H_1\|^2), \ t \geqslant 0, \tag{3.23}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the $H^{-1} \times L^2 \times L^2$-norm. □

It follows from Theorem 3.1 that we can define the family of solving operators

$$S(t) : \Phi_1 \to \Phi, (u_0, H_0, H_1) \to (u(t), H(t), \frac{\partial H(t)}{\partial t}), t \geqslant 0,$$

where

$$\Phi := \left\{ \left(u, H, \frac{\partial H}{\partial t} \right) \in H^1(\Omega)^3;\ |u| < 1 \text{ a.e.} \right\}$$

and

$$\Phi_1 := \Phi \cap \left\{ \left(u, H, \frac{\partial H}{\partial t} \right) \in L^\infty(\Omega) \times H^1(\Omega)^2;\ \|u\|_{L^\infty(\Omega)} < 1 \right\}.$$

We then deduce from (2.13) the

Theorem 3.3 *The semigroup $S(t)$ is dissipative in $H^1(\Omega)^3$, in the sense it possesses a bounded absorbing set $\beta_1 \subset H^1(\Omega)^3$ (i.e., $\forall B \subset \Phi_1$ bounded, $\exists t_0 = t_0(B)$ such that $t \geqslant t_0 \Rightarrow S(t)B \subset \beta_1$).*

We now assume that

$$\lim_{s \to \pm 1} F(s) = c, \tag{3.24}$$

where c is a constant (note that this holds for the thermodynamically relevant logarithmic potentials). Then, $S(t)$ is a semigroup now (i.e. $S(0) = I$ (identity operator) and $S(t + \tau) = S(t) \circ S(\tau),\ t, \tau \geqslant 0$) defined on Φ.

As a consequence of Theorem 3.3 and of (3.24), it follows from standard results (see, e.g. [3,29,42,46]) that we have the following theorem.

Theorem 3.4 *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ on Φ (i.e. $\mathcal{A}$ is compact in $H^{-1}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, bounded in Φ, invariant and attracts the images of all bounded subsets of Φ with respect to the topology of $H^{-1}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$).*

Remark 3.1 In order to prove that one has the global attractor and, in particular, the attraction property in the natural topology of the phase space Φ, one would need additional regularity on the solutions or the strict separation property from the singular values ± 1 which we are able to prove it in two dimensional space (see Section 5 below). We can also note that it follows from (3.23) that we can extend, in a unique way and by continuity, the semigroup $S(t)$ to the closure of Φ in the $H^{-1}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$-topology, namely to

$$\overline{\Phi} = \left\{ \left(u, H, \frac{\partial H}{\partial t} \right) \in L^\infty(\Omega) \times H^1(\Omega)^2;\ \|u\|_{L^\infty(\Omega)} \leqslant 1 \right\}.$$

The corresponding semigroup again possesses the global attractor which is precisely $\mathcal{A}$.

4 Further Regularity Results

In what follows, we set $V = H_0^1(\Omega)$. We also denote by V' its dual space and by $\|\cdot\|_{V'}$ its norm.

We can decompose the singular potential F as

$$F(x) = S(x) - \frac{\theta_0}{2} x^2,$$

with

$$\lim_{x\to -1} S'(x) = -\infty, \quad \lim_{x\to +1} S'(x) = +\infty, \quad S''(x) \geqslant \theta > 0, \quad \forall x \in (-1, 1) \tag{4.1}$$

and we let

$$\theta - \theta_0 = \alpha > 0. \tag{4.2}$$

We report here below a Trudinger–Moser type inequality (see, e.g., [44]) which will be needed later.

Lemma 4.1 *Let Ω be a bounded smooth domain of $\mathbb{R}^2$. Then, there exists a positive constant C such that*

$$\int_\Omega e^{|u|}\, dx \leqslant C e^{C\|u\|_V^2}, \quad \forall u \in V. \tag{4.3}$$

Let us now define the free energy functional

$$\mathcal{E}(u) = \frac{1}{2}\left(\|u\|^2_{H^1(\Omega)} + 2\int_\Omega F(u)\, dx + \|H\|^2 - 2((H, u)) + \left\|\frac{\partial H}{\partial t}\right\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1}\right). \tag{4.4}$$

Rewriting Eq. (1.16) in the equivalent form

$$\frac{\partial u}{\partial t} = \Delta \mu, \tag{4.5}$$

$$\mu = -\Delta u + u + F'(u) - H. \tag{4.6}$$

Theorem 4.1 *Let $u_0 \in V$ such that $F(u_0) \in L^1(\Omega)$. Then, there exists a unique solution $u \in \mathcal{C}([0, T], V)$ which fulfills the dissipative estimate*

$$\mathcal{E}(u(t), H(t), \partial_t H(t)) + \int_t^{t+1} \left\{ \|\nabla \mu(\tau)\|^2 + \left\|\frac{\partial H}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial H}{\partial t}(\tau)\right\|^2_{-1} \right\} d\tau \leqslant \mathcal{E}(u_0, H_0, H_1), \quad \forall t \geqslant 0. \tag{4.7}$$

Proof The existence and uniqueness of the solution can be proved in the same way as in section 3. Therefore, we confine ourselves only to the proof of (4.7).
We start by differentiating Eq. (1.16) with respect to time to find

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = \frac{\partial H}{\partial t}, \tag{4.8}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Gamma, \tag{4.9}$$

Multiplying (4.8) by $t\frac{\partial u}{\partial t}$, then using (1.25), we obtain

$$\frac{1}{2}\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}\right) + t\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)} \leqslant \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + c't\left\|\frac{\partial u}{\partial t}\right\|^2 + t\left(\left(\frac{\partial H}{\partial t}, \frac{\partial u}{\partial t}\right)\right). \tag{4.10}$$

Employing the interpolation inequality

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|^2 &\leqslant c'\left\|\frac{\partial u}{\partial t}\right\|_{-1}\left\|\nabla\frac{\partial u}{\partial t}\right\| \\ &\leqslant c'\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2,\end{aligned}$$

we find

$$\frac{1}{2}\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right)+ct\left\|\frac{\partial u}{\partial t}\right\|_{H^1(\Omega)}^2 \leqslant \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+c't\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial H}{\partial t}\right\|^2\right). \tag{4.11}$$

We deduce from (2.15)–(2.16) (which hold when $N \to +\infty$), (4.11), and Gronwall's lemma that

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2 \leqslant \frac{1}{t}Q(\|u_0\|_{H^1(\Omega)},\|H_0\|_{H^1(\Omega)},\|H_1\|_{H^1(\Omega)}),\ t\in(0,1]. \tag{4.12}$$

Multiplying (4.8) by $\dfrac{\partial u}{\partial t}$ and using (1.25) and an interpolation inequality, we have

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+c\left\|\frac{\partial u}{\partial t}\right\|_{H^1(\Omega)}^2 \leqslant c'\left(\left\|\frac{\partial H}{\partial t}\right\|^2+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\right). \tag{4.13}$$

Finally, we conclude from (4.12), (4.13) and Gronwall's lemma that

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2 \leqslant e^{ct}Q(\|u_0\|_{H^1(\Omega)},\|H_0\|_{H^1(\Omega)},\|H_1\|_{H^1(\Omega)}),\ t\geqslant 1, \tag{4.14}$$

where Q denotes a monotone increasing function.
Integrating now (4.13) between t and $t+1$ and using (4.14), we find

$$\int_t^{t+1}\left\|\frac{\partial u}{\partial t}\right\|_{H^1(\Omega)}^2 dt \leqslant e^{ct}Q(\|u_0\|_{H^1(\Omega)},\|H_0\|_{H^1(\Omega)},\|H_1\|_{H^1(\Omega)}), \tag{4.15}$$

for $t \geqslant 1$. Therefore, $\dfrac{\partial u}{\partial t}\in L^2(t,t+1;H^1(\Omega))$.
Then, Eq. (4.5) is equivalent to

$$((u_t,v))+((\nabla\mu,\nabla v))=0,\quad \forall v\in H_0^1(\Omega). \tag{4.16}$$

Using equation (4.16) and the standard chain rule in $L^2(0,T;V)\cap H^1(0,T;V')$, we get the energy equality

$$\frac{d}{dt}\mathcal{E}(u,H,\partial_t H)+\|\nabla\mu\|^2+\left\|\frac{\partial H}{\partial t}\right\|^2+\left\|\frac{\partial H}{\partial t}\right\|_{-1}^2=0$$

and it follows from the Gronwall lemma that

$$\mathcal{E}(u(t), H(t), \partial_t H(t)) + \int_t^{t+1} \left\{ \|\nabla\mu(\tau)\|^2 + \left\| \frac{\partial H}{\partial t}(\tau) \right\|^2 + \left\| \frac{\partial H}{\partial t}(\tau) \right\|_{-1}^2 \right\} d\tau \\ \leqslant \mathcal{E}(u_0, H_0, H_1), \ \forall t \geqslant 0. \tag{4.17}$$

□

In the sequel, according to (4.17), the generic positive constant C may also depend on the initial energy $\mathcal{E}(u_0)$. In particular, we will use

$$\mathcal{E}(u(t), H(t), \partial_t H(t)) + \int_t^{t+1} \left\{ \|\nabla\mu(\tau)\|^2 + \left\| \frac{\partial H}{\partial t}(\tau) \right\|^2 + \left\| \frac{\partial H}{\partial t}(\tau) \right\|_{-1}^2 \right\} d\tau \leqslant C, \ \ \forall t \geqslant 0. \tag{4.18}$$

Theorem 4.2 *Let the assumptions of Theorem 4.1 holds. Then, there exists a positive constant C such that*

$$\|\mu\|_{L^\infty(1,t;V)} \leqslant C, \ \ \forall t \geqslant 1 \tag{4.19}$$

and

$$\|u_t\|_{L^\infty(1,t;V')} + \|u_t\|_{L^2(t,t+1;V)} \leqslant C, \ \ \forall t \geqslant 1. \tag{4.20}$$

Proof Testing (4.5) by μ_t, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + ((u_t, \mu_t)) = 0. \tag{4.21}$$

We observe that

$$((u_t, \mu_t)) \geqslant \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u_t\|^2 - C\|u_t\|_{V'}^2 - C'. \tag{4.22}$$

Setting

$$\Psi(t) = \frac{1}{2}\|\nabla\mu(t)\|^2,$$

we end up with the differential inequality

$$\frac{d}{dt}\Psi + \frac{1}{2}\|u_t\|_{H^1(\Omega)}^2 \leqslant C\|u_t\|_{V'}^2 + C'. \tag{4.23}$$

Therefore, the uniform Gronwall lemma leads to

$$\Psi(t) \leqslant C, \ \ \forall t \geqslant 1.$$

In particular, we have the bound

$$\|\mu\|_{L^\infty(1,t;V)} \leqslant C, \ \ \forall t \geqslant 1,$$

which, in turns, gives

$$\|u_t\|_{L^\infty(1,t;V')} \leqslant C, \ \ \forall t \geqslant 1.$$

The desired conclusion, (4.20) follows from an integration in time of (4.23) on the time interval $(t, t+1),\ t \geqslant 1$, combined with the previous inequality. □

Remark 4.1 The proof of Theorem 4.2 is formal, but it can be justified within a Galerkin scheme as in the proof of Theorem 3.1. More precisely, all the computations can be regorously performed within the Galerkin scheme. Given that F'' is controlled from below, the estimated turn out to be independent of the approximation parameter and a final passage to the limit gives the result.

5 The Strict Separation Property in Two Dimensions

The main result of the paper reads as follows.

Theorem 5.1 *Let $n = 2$ and the assumptions of Theorem 4.1 hold. In addition, we require that S satisfies*

$$|S''(x)| \leqslant e^{C|S'(x)|+C}, \quad \forall x \in (-1, 1), \tag{5.1}$$

for some positive C and S'' is convex. Then, there exists $\delta > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant 1 - \delta, \quad \forall t \geqslant 2. \tag{5.2}$$

The proof is based on some technical lemmas and on additional assumptions on the singular part of F which are fulfilled, for instance, by the logarithmic free energy (1.20).

Lemma 5.1 *Let the assumptions of Theorem 5.1 hold. Then, for any $p \geqslant 1$, there exists a positive constant C depending on p such that*

$$\|S''(u)\|_{L^p(t,t+1;L^p(\Omega))} \leqslant C, \quad \forall t \geqslant 1. \tag{5.3}$$

Proof Equation (1.16) can be written in the equivalent form

$$-\Delta u + S'(u) = \tilde{\mu}, \tag{5.4}$$

where

$$\tilde{\mu} = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + H - u + \theta_0 u. \tag{5.5}$$

For any $L > 0$, we consider

$$g = S'(u)e^{L|S'(u)|}.$$

We observe that

$$\nabla g = S''(u)\left[1 + L|S'(u)|\right]e^{L|S'(u)|}\nabla u.$$

Then, we consider Eq. (5.4) and test it with g. This yields

$$\int_\Omega \nabla u \cdot \nabla u S''(u)\left[1 + L|S'(u)|\right]e^{L|S'(u)|}\,dx + \int_\Omega S'(u)S'(u)e^{L|S'(u)|}\,dx$$
$$= \int_\Omega \tilde{\mu} S'(u)e^{L|S'(u)|}\,dx.$$

Using the convexity of S'' with a generalized Young's inequality [1] and applying (4.3), we obtain that

$$\int_\Omega S'(u)^2 e^{L|S'(u)|}\,dx \leqslant C, \tag{5.6}$$

where C depends on L. On account of (5.1), we observe that, for any $p \geqslant 1$,

$$|S''(x)|^p \leqslant e^{pC}(C + |S'(x)|^2 e^{pC|S'(x)|}), \quad \forall x \in (-1, 1). \tag{5.7}$$

Combining finally (5.1) with (5.6) and (5.7), taking $L = pC$, we deduce that

$$\|S''(u)\|_{L^p(t,t+1;L^p(\Omega))} \leqslant C(p), \quad \forall t \geqslant 1.$$

□

Lemma 5.2 *Let the assumptions of Lemma 5.1 hold. Then, there exists a positive constant C such that*

$$\|u_t\|_{L^\infty(2,t;L^2(\Omega))} + \|u_t\|_{L^2(t,t+1;H^2(\Omega))} \leqslant C, \quad \forall t \geqslant 2. \tag{5.8}$$

Proof Differentiate equation (4.5) with respect to time, we obtain

$$\frac{\partial}{\partial t} u_t = \Delta \mu_t. \tag{5.9}$$

Multiplying (5.9) by u_t, we find

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \|\nabla u_t\|^2 + \|\Delta u_t\|^2 = \theta_0\|\nabla u_t\|^2 - ((H_t, \Delta u_t)) + \int_\Omega \frac{d}{dt}(S'(u))\cdot \Delta u_t\,dx. \tag{5.10}$$

We observe that,

$$|((H_t, \Delta u_t))| \leqslant \|H_t\|\|\Delta u_t\| \leqslant \frac{1}{4}\|\Delta u_t\|^2 + C$$

and

$$\int_\Omega \frac{d}{dt}(S'(u))\Delta u_t\,dx = \int_\Omega S''(u)u_t\Delta u_t\,dx \leqslant \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{4}\|\Delta u_t\|^2 + c\|S''(u)\|^4_{L^4(\Omega)}\|u_t\|^2.$$

Therefore, Eq. (5.10) becomes

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u_t\|^2 \leqslant c\|S''(u)\|^4_{L^4(\Omega)}\|u_t\|^2 + \theta_0\|\nabla u_t\|^2 + c, \tag{5.11}$$

where c depends on p. Furthermore, by interpolation

$$\theta_0\|\nabla u_t\|_{H^1(\Omega)} \leqslant \frac{1}{4}\|\Delta u_t\|^2 + c\|u_t\|^2, \quad c > 0.$$

Therefore, it follows

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{4}\|\Delta u_t\|^2\leqslant c(1+\|S''(u)\|^4_{L^4(\Omega)})\|u_t\|^2+c. \tag{5.12}$$

Using (5.3) and the uniform Gronwall lemma on (5.12), we obtain the desired result. □

We now have all the ingredients to show the strict separation property.

Proof of Theorem 5.1 We consider the elliptic equation (5.4). Due to Lemma 5.2 and the elliptic regularity, $\tilde{\mu}$ satisfies

$$\|\tilde{\mu}\|_{L^\infty(\Omega\times(2,t))}\leqslant C,\quad \forall t\geqslant 2.$$

Testing (5.4) by $|S'(u)|^{p-2}S'(u)$, we get

$$(p-1)\int_\Omega |S'(u)|^{p-2}S''(u)|\nabla u|^2\,dx+\|S'(u)\|^p_{L^p(\Omega)}=\int_\Omega \tilde{\mu}|S'(u)|^{p-2}S'(u)\,dx.$$

Noting that the first term is non-negative, integrating in time from t to $t+1$ and an application of the Holder inequality, we have for all $t\geqslant 2$

$$\|S'(u)\|_{L^p(\Omega\times(t,t+1))}\leqslant\|\tilde{\mu}\|_{L^p(\Omega\times(t,t+1))}\leqslant C\|\tilde{\mu}\|_{L^\infty(\Omega\times(t,t+1))}\leqslant C,$$

where C is independent of p and t. Applying Theorem 2.14 in [1], we obtain

$$\|S'(u)\|_{L^\infty(\Omega\times(t,t+1))}\leqslant C,\quad \forall t\geqslant 2.$$

This implies that there exists $\delta>0$ such that

$$\|u\|_{L^\infty(\Omega\times(t,t+1))}\leqslant 1-\delta,\quad \forall t\geqslant 2.$$

Since u belongs to $L^\infty(0,t;L^\infty(\Omega))$ for all $t\geqslant 0$, we also infer that

$$\|u\|_{L^\infty(2,t;L^\infty(\Omega))}\leqslant 1-\delta,\quad \forall t\geqslant 2.$$

Finally, we deduce (5.2) from the continuity in time. □

6 Discretization of the Conserved Caginalp Phase-Field System

We note that the system (1.16)-(1.17) is equivalent to the following system :

$$\begin{aligned}
\frac{\partial u}{\partial t}-\Delta\mu&=0,\\
-\varepsilon\Delta u+u+\frac{1}{\varepsilon}f(u)-H&=\mu,\\
(I_d-\Delta)\left(\tau\frac{\partial^2 H}{\partial t^2}+\frac{\partial H}{\partial t}\right)-\Delta H&=-\Delta u,\\
u|_{t=0}=f_u(x,y,0),\quad H|_{t=0}=f_H(x,y,0),&\quad \frac{\partial H}{\partial t}|_{t=0}=g_H(x,y,0),
\end{aligned} \tag{6.1}$$

where, in this section, we present the spatial discretization of (6.1) using finite element method with $\mathbb{P}_1$ continuous piecewise linear functions and a first-order semi-implicit scheme for the time marching scheme.

6.1 Spatial Discretization

We let Ω be a convex, planar domain and $\mathbf{T}_h$ be a regular, quasi uniform triangulation of Ω with triangles of maximum size $h < 1$. Setting $V_h = \{v_h \in C^0(\bar{\Omega}); v_h|_{\mathbf{T}_h} \in \mathbb{P}_1(\mathbf{T}_h), \forall T \in \mathbf{T}_h\}$ a finite-dimensional subspace of $H^1(\Omega)$ where $\mathbb{P}_1$ is the set of all polynomials of degrees ≤ 1 with real coefficients and we consider the weak formulation of (6.1):
Find $u_h, \mu_h, H_h \in V_h$ such that, $\forall \phi_h \in V_h$,

$$\begin{aligned} \left(\left(\frac{\partial u_h}{\partial t} - \Delta\mu_h, \phi_h\right)\right) &= 0, \\ \left(\left(-\varepsilon\Delta u_h + u_h + \frac{1}{\varepsilon} f(u_h) - H_h, \phi_h\right)\right)) &= \left(\left(\mu_h, \phi_h\right)\right), \\ \left(\left((I_d - \Delta)\left(\tau\frac{\partial^2 H_h}{\partial t^2} + \frac{\partial H_h}{\partial t}\right) - \Delta H_h, \phi_h\right)\right) &= -\left(\left(\Delta u_h, \phi_h\right)\right), \\ u_h|_{t=0} = f_{u_h(x,y,0)}, \quad H_h|_{t=0} = f_{H_h(x,y,0)}, \quad & \frac{\partial H_h}{\partial t}|_{t=0} = g_{H_h(x,y,0)}. \end{aligned} \tag{6.2}$$

6.2 Time Marching Scheme

We discretize system (6.2) in time using a first-order semi-implicit scheme. To this end, let us denote by $(u_h^{n+1}, \mu_h^{n+1}, H_h^{n+1})$ and (u_h^n, μ_h^n, H_h^n) the approximate value at time $t = t^{n+1}$ and $t = t^n$, respectively, and by δt the time step. Then, owing to (6.2), the unknown fields at time $t = t^{n+1}$ are defined as the solution of:

$$\begin{aligned} &\left(\left(u_h^{n+1} - \delta t \cdot \Delta\mu_h^{n+1}; \phi_h\right)\right) = \left(\left(u_h^n; \phi_h\right)\right); \\ &\left(\left((\mu_h^{n+1} - (Id - \varepsilon\Delta)u_h^{n+1} + H_h^{n+1}; \phi_h\right)\right) = \left(\left(\frac{1}{\varepsilon} \cdot f(u_h^n); \phi_h\right)\right), \\ &\left(\left(\left((\tau + \delta t)\,(I_d - \Delta) - \delta t^2 \cdot \Delta\right)H_h^{n+1} + \delta t^2 \Delta u_h^{n+1}; \phi_h\right)\right) \\ &\qquad = \left(\left((2\tau + \delta t) \cdot (I_d - \Delta)\, H_h^n - \tau\,(I_d - \Delta)\, H_h^{n-1}; \phi_h\right)\right), \\ &u_h^n|_{t=0} = f_{u_h(x,y,0)}, \quad H_h^{n-1}|_{t=0} = f_{H_h(x,y,0)}, \quad H_h^n|_{t=0} = H_h^{n-1}|_{t=0} + \delta t \cdot g_{H_h(x,y,0)}, \end{aligned}$$

in which (6.3) can be written equivalently in the following matrix form ($\mathbf{AX} = \mathbf{B}$):

$$\begin{pmatrix} I_d & -\delta t\Delta(\cdot) & 0 \\ -(I_d - \varepsilon\Delta)(\cdot) & I_d & I_d \\ \delta t^2\Delta(\cdot) & 0 & \left((\tau + \delta t)\,(I_d - \Delta) - \delta t^2 \cdot \Delta\right)(\cdot) \end{pmatrix} \begin{pmatrix} u_h^{n+1} \\ \mu_h^{n+1} \\ H_h^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{F}(u_h^n) \\ \mathbf{G}(u_h^n) \\ \mathbf{H}(H_h^n, H_h^{n-1}), \end{pmatrix} \tag{6.3}$$

where

$$\begin{aligned}\mathbf{F}(u_h^n) &= u_h^n\\ \mathbf{G}(u_h^n) &= \frac{1}{\varepsilon}\cdot f(u_h^n);\\ \mathbf{H}(H_h^n, H_h^{n-1}) &= (2\tau+\delta t)\cdot(I_d-\Delta)\,H_h^n - \tau\,(I_d-\Delta)\,H_h^{n-1}.\end{aligned} \tag{6.4}$$

7 Numerical Simulations

We perform several numerical simulations using the FreeFem++ software [23], comparing the conserved Caginalp phase-field system (6.3)–(6.4) with the cubic nonlinear term $f(s) = s^3 - .8s$ and the logarithmic one $f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\frac{1+s}{1-s}\right)$ when $(\kappa_0, \kappa_1) = (\ln(3), 0.8)$. We compute the propagation in a square $[0, 1] \times [0, 1]$ of an initial random function for u between $[-1, 1]$ and a constant enthalpy $H = .1$ and we take $f_H(x, y, 0) = .1,\ g_H(x, y, 0) = 0$ and a periodic boundary condition for u and H.

We will consider three cases for the space discretization, $\delta x \in \left\{\frac{1}{32}, \frac{1}{64}, \frac{1}{128}\right\}$, and we take $\varepsilon = \delta x$ and $\delta t = \varepsilon^3$.

In order to compare the conserved Caginalp model for both potentials, we plot in the following Figs. 2, 3, 4, 5, 6 and 7 the solutions u, H in 2D, a cut of the solutions for $x = .5$ or $y = .5$, the minima and maxima of u and H during the simulation, their corresponding mass conservation and energy with different values of δx. We also use the parallelization mpi of our script using Petsc [26] in order to converge rapidly to the final solution; thus, for $\delta x = 1/32$, we use 3 processors, for $\delta x = 1/64$, we use 8 processors and, for $\delta x = 1/128$, we use 11 processors.

As far as the Cahn–Hilliard equation is concerned, it is known that its solution converges to one steady state solution which is either a straight line between -1 and 1 or a circle between -1 and 1.

Interestingly, in our case, we instead observe a time periodic solution for u, due to the influence of the enthalpy H. More precisely, when the solution u first converges to its steady state solution, either circle or straight line, u and H then start to propagate in the same direction, either from right to left as in the case of Fig. 3, downward as in the case of Fig. 7, right to left as in the case of Fig. 2, left to right as in the case of Fig. 6 or, when we have a circular steady state, from top left to lower right as in Fig. 5 and from lower right to top left as in Fig. 4.

We can see that we indeed have the mass conservation for u in all cases and the energy decays until the solution converges to the steady state and then starts to oscillate due to the time periodic solution.

We can see that the solution u converges to $-.9$ and $.9$ for the polynomial potential and converges to -0.8 and 0.8 for the logarithmic potential; the enthalpy H has a similar behavior for both potentials, whereas, concerning the energy, we do not obtain same values, due to the fact that the maximum of the double-well potential differs between the logarithmic potential 1.1 and the polynomial one .1.

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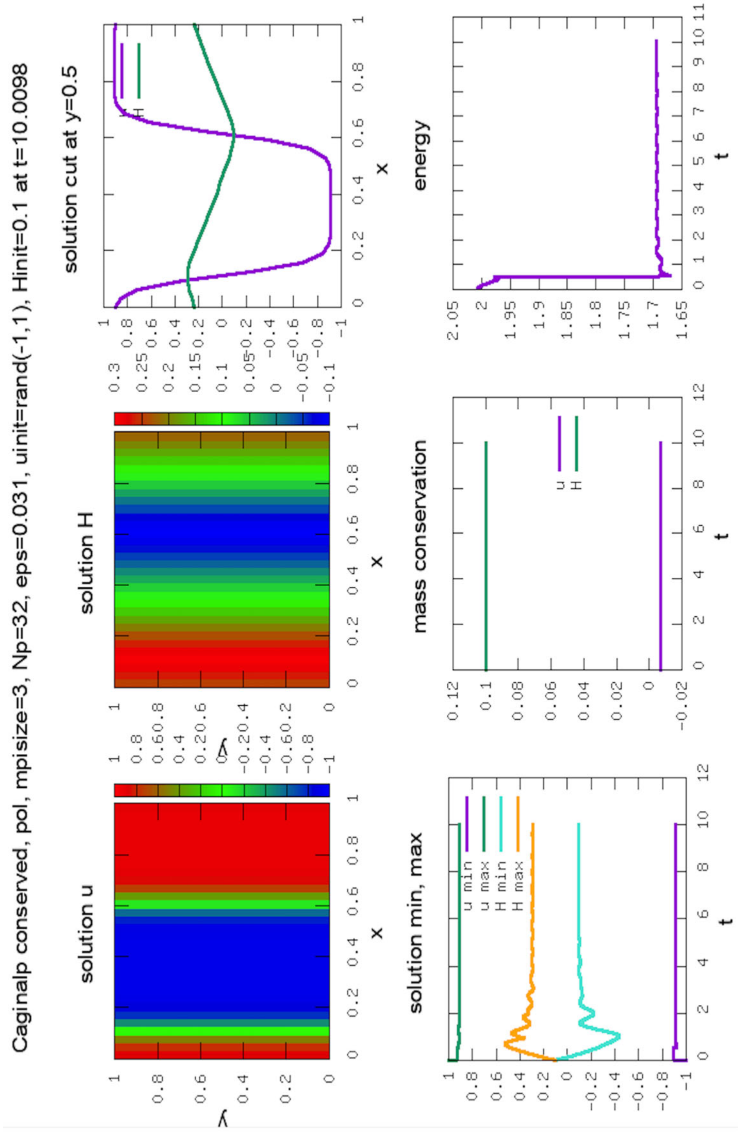

Fig. 2 Solution goes from right to left, polynomial potential, $\delta x = 1/32$

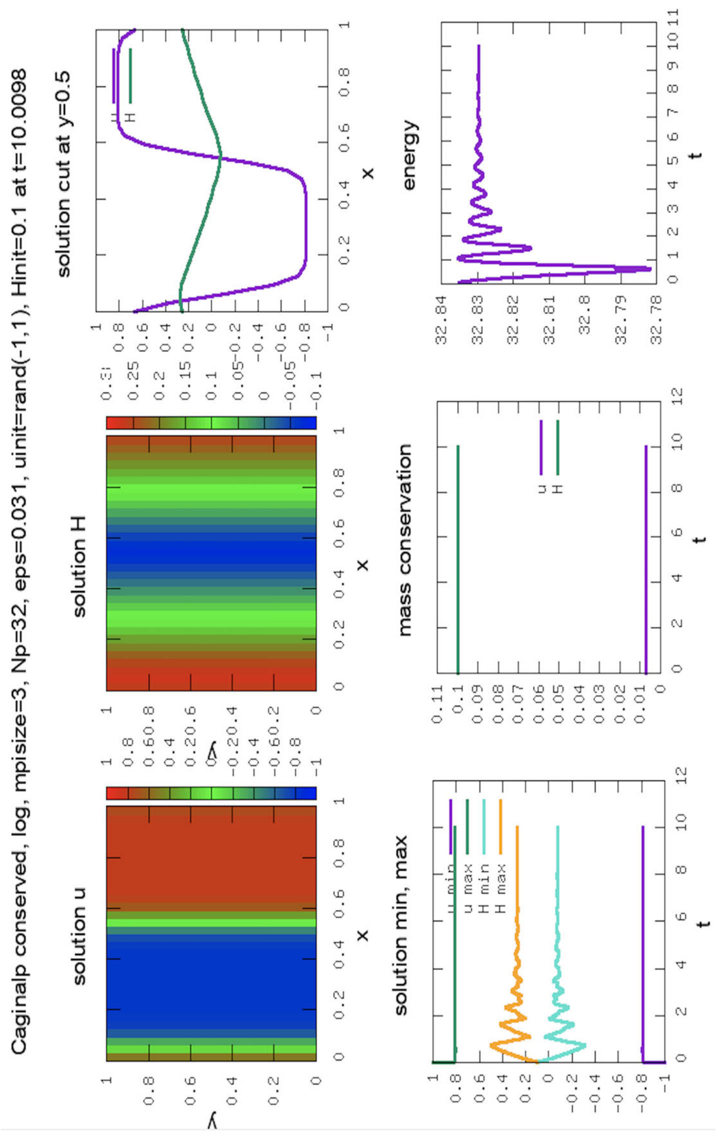

Fig. 3 Solution goes from right to left, logarithmic potential, $\delta x = 1/32$

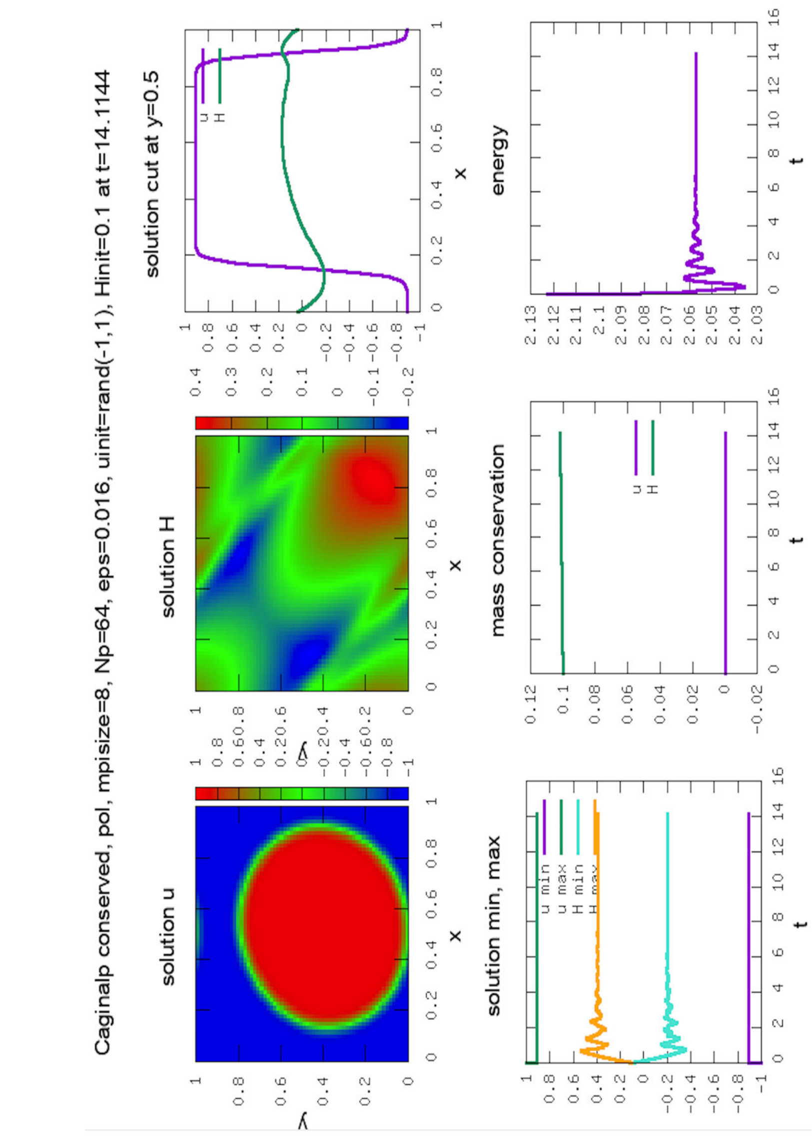

Fig. 4 Solution goes from lower right to top left, polynomial potential, $\delta x = 1/64$

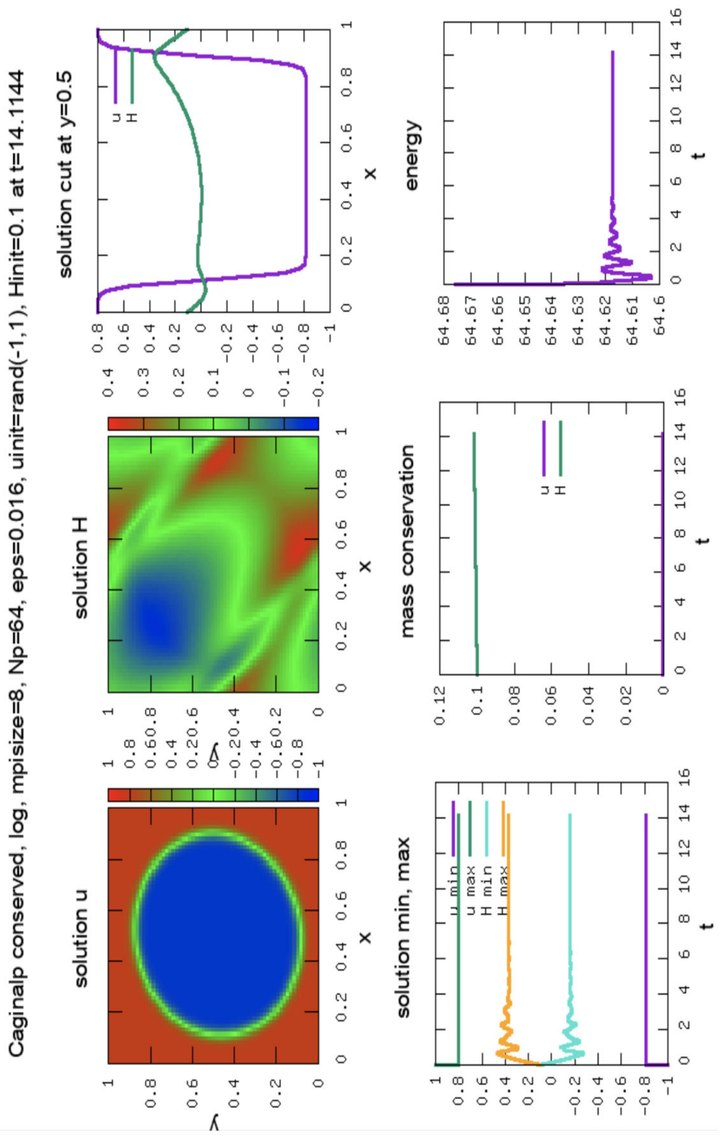

Fig. 5 Solution goes from top left to lower right, logarithmic potential, $\delta x = 1/64$

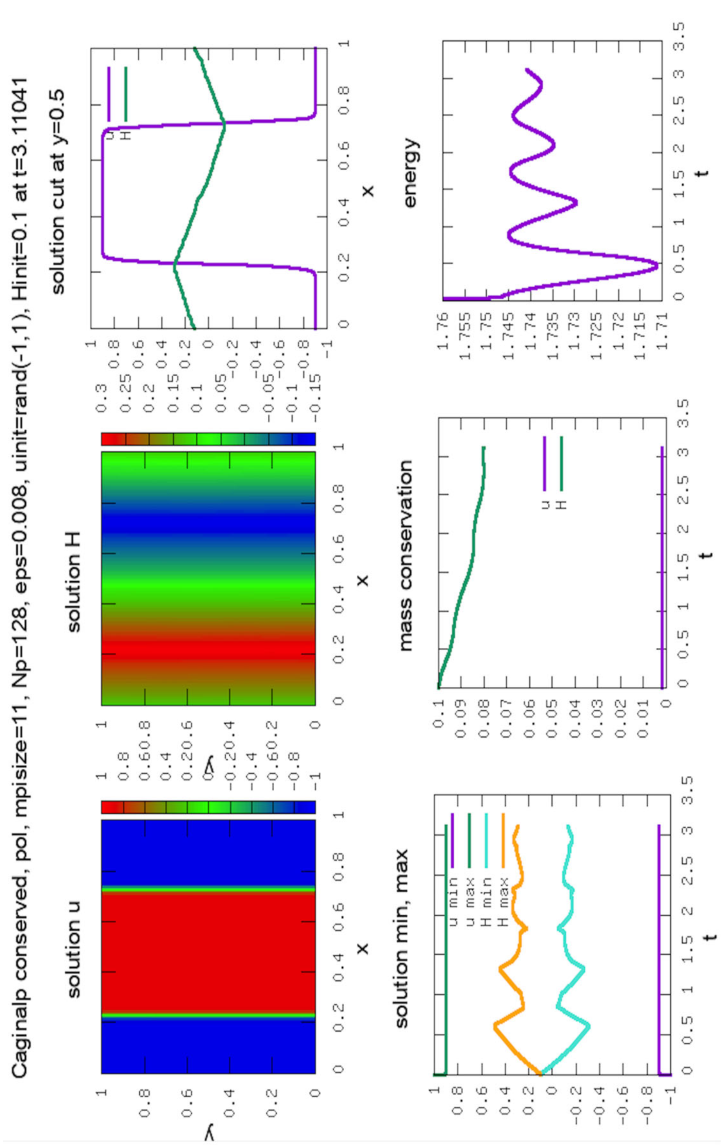

Fig. 6 Solution goes from left to right, polynomial potential, $\delta x = 1/128$

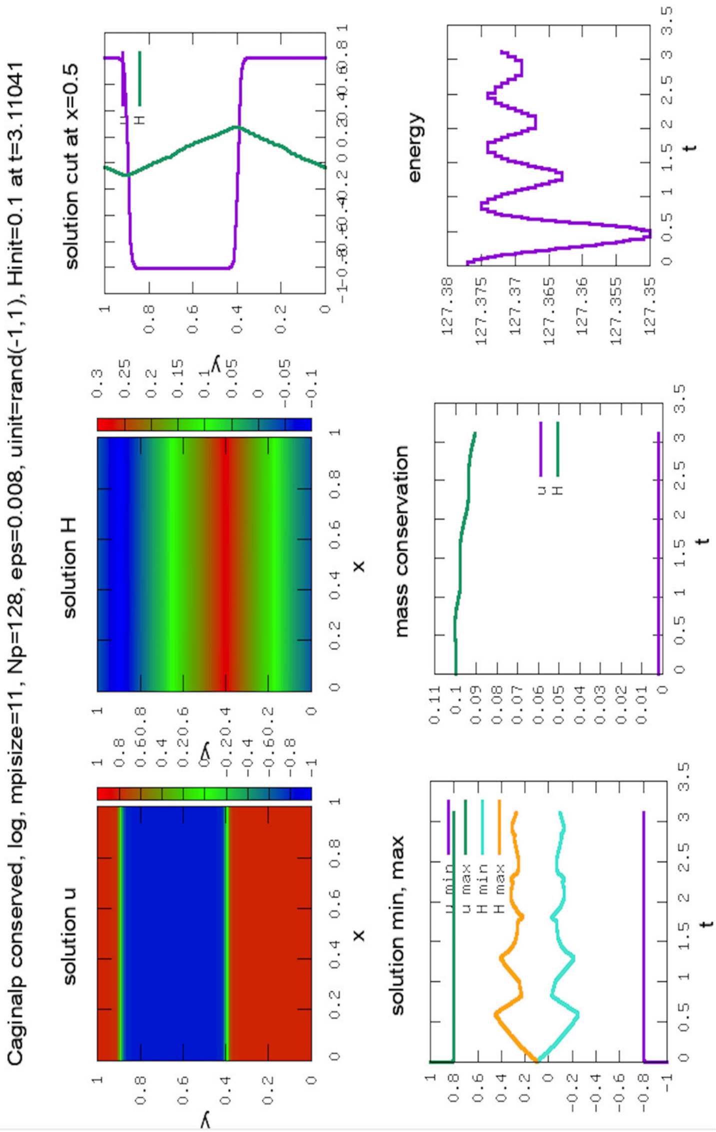

Fig. 7 Solution goes downward, logarithmic potential, $\delta x = 1/128$

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