DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **24**, Number **3**, March **2019**

doi:10.3934/dcdsb.2019019

pp. **X–XX**

ON THE NONCONSERVED CAGINALP PHASE-FIELD SYSTEM BASED ON THE MAXWELL-CATTANEO LAW WITH TWO TEMPERATURES AND LOGARITHMIC POTENTIALS

Ahmad Makki

Université de Poitiers
Laboratoire de Mathématiques et Applications
UMR CNRS 7348, SP2MI
Boulevard Marie et Pierre Curie - Téléport 2
F-86962 Chasseneuil Futuroscope Cedex, France

Alain Miranville*

Xiamen University, School of Mathematical Sciences
Fujian Provincial Key Laboratory of Mathematical Modeling
and High Performance Scientific Computing
Xiamen, Fujian, China
and
Université de Poitiers
Laboratoire de Mathématiques et Applications, UMR CNRS 7348
Boulevard Marie et Pierre Curie - Téléport 2
F-86962 Chasseneuil Futuroscope Cedex, France

Georges Sadaka

Université de Picardie Jules Verne
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
UMR CNRS 7352
Pôle Scientifique, 33, rue Saint Leu
F-80039 Amiens, France

Abstract. Our aim in this article is to study generalizations of the nonconserved Caginalp phase-field system based on the Maxwell-Cattaneo law with two temperatures for heat conduction and with logarithmic nonlinear terms. We obtain well-posedness results and study the asymptotic behavior of the system. In particular, we prove the existence of the global attractor. Furthermore, we give some numerical simulations, obtained with the FreeFem++ software [24], comparing the nonconserved Caginalp phase-field model with regular and logarithmic nonlinear terms.

1. Introduction.

The nonconserved Caginalp phase field system

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T, \tag{1}$$

$$\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t}, \tag{2}$$

2010 *Mathematics Subject Classification.* 35B45, 35K55, 35L15.
Key words and phrases. Caginalp system, Maxwell-Cattaneo law, two temperatures, logarithmic nonlinear terms, well-posedness, dissipativity, global attractor, simulations.

* Corresponding author: Alain Miranville.

has been proposed in [5] to model phase transition phenomena, such as melting-solidification phenomena. Here, u is the order parameter, T is the relative temperature (defined as $T = \tilde{T} - T_E$, where $\tilde{T}$ is the absolute temperature and T_E is the equilibrium melting temperature) and f is the derivative of a double-well potential F (a typical choice is $F(s) = \dfrac{1}{4}(s^2-1)^2$, hence the usual cubic nonlinear term $f(s) = s^3 - s$). Furthermore, here and below, we set all physical parameters equal to one. This system has been extensively studied; we refer the reader to, e.g., [1], [2], [3], [4], [10], [11], [13], [17], [18], [19], [20], [21], [22], [27], [35] and [37].

These equations can be derived as follows. One introduces the (total Ginzburg-Landau) free energy

$$\Psi = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - uT - \frac{1}{2}T^2\right) dx, \tag{3}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^n, n = 1, 2$ or 3, with boundary Γ) and the enthalpy

$$H = u + T. \tag{4}$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = -\frac{D\Psi}{Du}, \tag{5}$$

where $\dfrac{D}{Du}$ denotes a variational derivative with respect to u, which yields (1). Then, we have the energy equation

$$\frac{\partial H}{\partial t} = -\operatorname{div} q, \tag{6}$$

where q is the heat flux. Assuming finally the usual Fourier law for heat conduction,

$$q = -\nabla T, \tag{7}$$

we obtain (2).

Now, one essential drawback of the Fourier law is that it predicts that thermal signals propagate at an infinite speed, which violates causality (the so-called paradox of heat conduction, see [14]). To overcome this drawback, or at least to account for more realistic features, several alternatives to the Fourier law, based, for example, on the Maxwell-Cattaneo law or recent laws from thermomechanics, have been proposed and studied in, e.g., [25], [26], [28], [29], [30], [31] and [32].

In the late 1960's, several authors proposed a heat conduction theory based on two temperatures (see [7], [8] and [9]). More precisely, one now considers the conductive temperature T and the thermodynamic temperature θ. For time-independent problems the difference between these temperatures is proportional to the heat supply; they thus coincide when there is no heat supply. However, for time-dependent problems, they are generally different even in the absence of heat supply: this is in particular the case for non-simple materials. In that case, the two temperatures are related as follows:

$$\theta = T - \Delta T. \tag{8}$$

The nonconserved Caginalp system was studied in [15] for the classical Fourier law with two temperatures and in [33] for the type III thermomechanics theory [23] with two temperatures recently proposed in [37] (see also [16]).

In this article, we consider the theory of two-temperature-generalized thermoelasticity proposed in [39] and based on the Maxwell-Cattaneo law.

In that case, the free energy reads, in terms of the (relative) thermodynamic temperature θ,

$$\Psi = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2 \right) dx \tag{9}$$

and (5) yields, in view of (8), the following evolution equation for the order parameter:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = T - \Delta T. \tag{10}$$

Furthermore, to obtain the corresponding generalized heat equation, one writes

$$\frac{\partial H}{\partial t} = -\operatorname{div} q, \tag{11}$$

$$H = u + \theta = u + T - \Delta T, \tag{12}$$

where the heat flux q satisfies the Maxwell-Cattaneo law [39],

$$q + \tau \frac{\partial q}{\partial t} = -\nabla T, \tau > 0. \tag{13}$$

In particular, it follows from (11) that

$$\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} = -\operatorname{div} \left(q + \tau \frac{\partial q}{\partial t} \right),$$

hence, in view of (13),

$$\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} = \Delta T. \tag{14}$$

We thus deduce from (12) and (14) the generalized heat equation

$$(I - \Delta) \left(\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} \right) - \Delta T = -\tau \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}. \tag{15}$$

Here, the presence of the second derivative $\frac{\partial^2 u}{\partial t^2}$ makes the mathematical analysis of the equation particularly difficult and, to overcome such a difficulty, we will rewrite the equation in a different way, keeping the enthalpy H as unknown. Indeed, it follows from (12) and (14) that

$$(I - \Delta) \left(\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} \right) = \Delta(T - \Delta T),$$

hence

$$(I - \Delta) \left(\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t} \right) - \Delta H = -\Delta u. \tag{16}$$

Furthermore, owing again to (12), (10) can be written as

$$\frac{\partial u}{\partial t} - \Delta u + u + f(u) = H. \tag{17}$$

In [34], the authors studied the well-posedness of the nonconserved Caginalp system (16)-(17), for regular nonlinear terms f and Dirichlet boundary conditions. It is however important to note that, in phase transition, regular nonlinear terms actually are approximations of thermodynamically relevant logarithmic ones of the form

$$f(s) = -2\kappa_0 s + \kappa_1 \ln \left(\frac{1+s}{1-s} \right), \tag{18}$$

with $s \in (-1, 1)$ and $0 < \kappa_1 < \kappa_0$, which follow from a mean-field model (see [6], [27]; in particular, the logarithmic terms correspond to the entropy of mixing).

In order to compare the logarithmic potentials with the cubic ones in the numerical simulations that we will perform, we will choose a cubic polynomial which has the same extrema as the logarithmic potential. To this end, we will consider the following cubic nonlinear terms:

$$f(s) = \begin{cases} .83(s^3 - .5^2 s), & \text{when} \quad (\kappa_0, \kappa_1) = (\ln(3), 1), & (19) \\ 2.5(s^3 - .9315^2 s), & \text{when} \quad (\kappa_0, \kappa_1) = (\ln(6), 1). & (20) \end{cases}$$

The original Caginalp phase-field system, with the aforementioned logarithmic nonlinear terms, was studied in [27]; see also [19] for a more general Caginalp phase-field system, with a nonlinear coupling between u and T.

In this article, we consider the nonconserved phase-field model (16)-(17), with the logarithmic nonlinear terms (18). The article is organized as follows. In Section 2, we derive *a priori* estimates which are of fundamental significance for what follows. In Section 3, we prove that the solutions are separated from the singular points of f, which allows us to prove the existence of global (in time) solutions. In Section 4, we study the dissipativity of the associated dynamical system. In Section 5, we prove the existence of the global attractor. In Section 6, we write the spatial and time discretizations of (16)-(17), which allows us finally, in Section 7, to give a comparison for the nonconserved Caginalp model with regular and the logarithmic nonlinear terms, first, by comparing the convergence rate of our codes and then by computing the propagation of a cross function for u and a constant enthalpy H. In particular, we give an example for which both potentials are comparable (this is expected when the quench is shallow, i.e., when κ_1 is close to κ_0) and a second one for which the logarithmic potential gives much better results.

Notation. We denote by $((\cdot\,,\,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. More generally, $\|\cdot\|_X$ denotes the norm in the Banach space X.

Throughout the article, the same letter c, c' (and, sometimes, c'', C) denotes (generally positive) constants which may vary from line to line. Similarly, the same letter Q denotes (positive) monotone increasing (with respect to each argument) functions which may vary from line to line.

Setting the problem. We consider the following initial and boundary value problem, in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$ or 3, with boundary Γ:

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + u + \frac{1}{\varepsilon} f(u) = H, \tag{21}$$

$$(I - \Delta)\left(\tau \frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t}\right) - \Delta H = -\Delta u, \tag{22}$$

$$u = H = 0 \quad \text{on} \quad \Gamma, \tag{23}$$

$$u|_{t=0} = u_0, \; H|_{t=0} = H_0, \; \frac{\partial H}{\partial t}|_{t=0} = H_1. \tag{24}$$

For simplicity, we set τ and ε equal to one in what follows. The nonlinear term f is defined as

$$f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\frac{1+s}{1-s}\right), \tag{25}$$

with $s \in]-1,1[$ and $0 < \kappa_1 < \kappa_0$. We then have

$$f'(s) = \frac{2\kappa_1}{1-s^2} - 2\kappa_0. \tag{26}$$

Lemma 1.1. *The nonlinear term f in (25) is of class $\mathcal{C}^\infty$ and satisfies*

$$-c_0 \leqslant F(s) \leqslant f(s)s + c_0, \quad c_0 \geqslant 0, \tag{27}$$

where $F(s) = \int_0^s f(\tau)\,d\tau$, and

$$f(0) = 0, \quad f'(s) \geqslant -c_1, \ c_1 \geqslant 0. \tag{28}$$

Proof. We have, for $s \in (-1,1)$,

$$\begin{aligned} F(s) = \int_0^s f(\tau)\,d\tau \quad &= -\kappa_0 s^2 + \kappa_1[s\ln(\frac{1+s}{1-s}) + \ln((1-s)(1+s))] \\ &= f(s)s + \kappa_0 s^2 + \kappa_1 \ln((1-s)(1+s)). \end{aligned}$$

Note that, for $s \in (-1,1)$,

$$\kappa_1 \ln((1-s)(1+s)) \leqslant 0$$

and

$$\kappa_1[s\ln(\frac{1+s}{1-s}) + \ln((1-s)(1+s))] \geqslant 0.$$

Therefore, we obtain

$$-c_0 \leqslant F(s) \leqslant f(s)s + c_0,$$

with $c_0 = \kappa_0 s^2 > 0$. Finally, it easily follows from (26) that

$$f'(s) \geqslant -c_1,$$

with $c_1 = 2\kappa_0 > 0$. □

Remark 1.1. *We can also endow the problem with periodic or Neumann boundary conditions. In these cases, we have, integrating (22) over Ω,*

$$\frac{d}{dt}(\frac{d\langle H\rangle}{dt} + \langle H\rangle) = 0, \tag{29}$$

where $\langle\cdot\rangle$ denotes the spatial average, which yields

$$\frac{d\langle H\rangle}{dt} + \langle H\rangle = \langle H_0 + H_1\rangle \tag{30}$$

and

$$\langle H(t)\rangle = \langle H_0 + H_1\rangle - \langle H_1\rangle e^{-t}, \quad t \geqslant 0. \tag{31}$$

Taking (29)-(31) into account, we can adapt the proofs below and derive the same well-posedness results. Note however that, in order to study the existence of attractors, we need to assume that

$$|\langle H_0 + H_1\rangle| \leqslant M_1, \quad |\langle H_1\rangle| \leqslant M_2. \tag{32}$$

It thus follows from (31) that

$$|(\frac{d\langle H\rangle}{dt} + \langle H\rangle)(t)| \leqslant M_1, \quad |\frac{d\langle H\rangle}{dt}(t)| \leqslant M_2, \quad |\langle H(t)\rangle| \leqslant M_1 + M_2, \quad \forall t \geqslant 0. \tag{33}$$

We can then define the family of solving operators

$$S(t) : \Phi_M \to \Phi_M, \quad (u_0, H_0, H_1) \mapsto (u(t), H(t), \frac{\partial H}{\partial t}(t)), \quad t \geq 0,$$

where

$$\Phi_M(=\Phi_{M_1,M_2}) = \{(\varphi,\theta,\xi) \in H^2(\Omega)^3;\ \|\varphi\|_{L^\infty(\Omega)} < 1,\ |\langle\theta+\xi\rangle| \leqslant M_1,\ |\langle\xi\rangle| \leqslant M_2\}.$$

This family of operators forms a semigroup which is continuous for the $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$-topology. We refer the interested reader to [33] for more details on the necessary modifications.

2. A priori estimates.

Remark 2.1. *We will make formal calculations here, keeping in mind that all these calculations can be rigorously justified by approaching the singular function f by regular functions of class $\mathcal{C}^1$. We will assume a priori that $\|u\|_{L^\infty((0,T)\times\Omega)} < 1$ and that $\|u_0\|_{L^\infty} < 1$.*

We multiply (21) by $\frac{\partial u}{\partial t}$ and have, integrating over Ω and by parts,

$$\frac{d}{dt}\left(\|u\|^2_{H^1(\Omega)} + 2\int_\Omega F(u)\,dx\right) + 2\left\|\frac{\partial u}{\partial t}\right\|^2 = 2\left(\left(H, \frac{\partial u}{\partial t}\right)\right), \tag{34}$$

noting that $\|\cdot\|^2_{H^1(\Omega)} = \|\cdot\|^2 + \|\nabla\cdot\|^2$.

We then multiply (22) by $(-\Delta)^{-1}\frac{\partial H}{\partial t}$ to obtain

$$\frac{d}{dt}\left(\|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right) + 2\left(\left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right) = 2\left(\left(u, \frac{\partial H}{\partial t}\right)\right). \tag{35}$$

Noting that

$$\left(\left(H, \frac{\partial u}{\partial t}\right)\right) = \frac{d}{dt}((u,H)) - \left(\left(u, \frac{\partial H}{\partial t}\right)\right),$$

we finally find, summing (34) and (35),

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|u-H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right)&\\ + 2\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right) &= 0.\end{aligned} \tag{36}$$

Next, we multiply (21) by u and have, owing to (27),

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2_{H^1(\Omega)} + c\int_\Omega F(u)\,dx \leqslant 2((H,u)) + c'. \tag{37}$$

Multiplying then (22) by $(-\Delta)^{-1}H$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\|H\|^2_{-1} + \|H\|^2 + 2\left(\left(\frac{\partial H}{\partial t}, H\right)\right)_{-1} + 2\left(\left(\frac{\partial H}{\partial t}, H\right)\right)\right) + 2\|H\|^2&\\ = 2((H,u)) + 2\left(\left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right)&.\end{aligned} \tag{38}$$

Summing (37) and (38), we find

$$\frac{d}{dt}\Big(\|u\|^2+\|H\|_{-1}^2+\|H\|^2+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)_{-1}+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)\Big)+c\Big(\|u-H\|^2 + \|\nabla u\|^2+2\int_\Omega F(u)\,dx\Big)\leqslant 2\Big(\Big\|\frac{\partial H}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial H}{\partial t}\Big\|^2\Big)+c',\ c>0. \tag{39}$$

Summing finally (36) and δ_1 times (39), where $\delta_1>0$ is chosen small enough, we have a differential inequality of the form

$$\frac{d}{dt}E_1+c\Big(E_1+\Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\leqslant c',\ c>0, \tag{40}$$

where

$$E_1=\|\nabla u\|^2+2\int_\Omega F(u)\,dx+\|u-H\|^2+\Big\|\frac{\partial H}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial H}{\partial t}\Big\|^2 + \delta_1\Big(\|u\|^2+\|H\|_{-1}^2+\|H\|^2+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)_{-1}+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)\Big) \tag{41}$$

satisfies

$$E_1\geqslant c\Big(\|u\|_{H^1(\Omega)}^2+\int_\Omega F(u)\,dx+\|H\|^2+\Big\|\frac{\partial H}{\partial t}\Big\|^2\Big)-c',\ c>0. \tag{42}$$

We now multiply (22) by $\frac{\partial H}{\partial t}$ to obtain

$$\frac{d}{dt}\Big(\|\nabla H\|^2+\Big\|\frac{\partial H}{\partial t}\Big\|_{H^1(\Omega)}^2\Big)+\Big\|\frac{\partial H}{\partial t}\Big\|_{H^1(\Omega)}^2\leqslant\|\nabla u\|^2. \tag{43}$$

Multiplying also (22) by H, we find

$$\frac{d}{dt}\Big(\|H\|_{H^1(\Omega)}^2+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)+2\Big(\Big(\nabla\frac{\partial H}{\partial t},\nabla H\Big)\Big)\Big)+\|\nabla H\|^2 \leqslant\|\nabla u\|^2+2\Big\|\frac{\partial H}{\partial t}\Big\|_{H^1(\Omega)}^2. \tag{44}$$

Summing (40), δ_2 times (43) and δ_3 times (44), where $\delta_2,\ \delta_3>0$ are chosen small enough, we have a differential inequatity of the form

$$\frac{d}{dt}E_2+c\Big(E_2+\Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\leqslant c',\ c>0, \tag{45}$$

where

$$E_2=E_1+\delta_2\Big(\|\nabla H\|^2+\Big\|\frac{\partial H}{\partial t}\Big\|_{H^1(\Omega)}^2\Big) + \delta_3\Big(\|H\|_{H^1(\Omega)}^2+2\Big(\Big(\frac{\partial H}{\partial t},H\Big)\Big)+2\Big(\Big(\nabla\frac{\partial H}{\partial t},\nabla H\Big)\Big)\Big) \tag{46}$$

satisfies

$$E_2\geqslant c\Big(\|u\|_{H^1(\Omega)}^2+\int_\Omega F(u)\,dx+\|H\|_{H^1(\Omega)}^2+\Big\|\frac{\partial H}{\partial t}\Big\|_{H^1(\Omega)}^2\Big)-c',\ c>0. \tag{47}$$

Gronwall's lemma implies that

$$u, H, \frac{\partial H}{\partial t} \in L^\infty(0,T,H^1(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^2(0,T,L^2(\Omega)).$$

We finally multiply (21) by $-\Delta u$ and obtain, owing to (28) and classical elliptic regularity results,

$$\frac{d}{dt}\|\nabla u\|^2 + c\|u\|^2_{H^2(\Omega)} \leqslant c'(\|\nabla u\|^2 + \|H\|^2),\ c > 0. \tag{48}$$

Summing (45) and δ_4 times (48), where $\delta_4 > 0$ is chosen small enough, we find a differential inequality of the form

$$\frac{dE_3}{dt} + c\left(E_3 + \|u\|^2_{H^2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2\right) \leqslant c',\ c > 0, \tag{49}$$

where

$$E_3 = E_2 + \delta_4\|\nabla u\|^2 \tag{50}$$

satisfies

$$E_3 \geqslant c\left(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u)\,dx + \|H\|^2_{H^1(\Omega)} + \left\|\frac{\partial H}{\partial t}\right\|^2_{H^1(\Omega)}\right) - c',\ c > 0. \tag{51}$$

Thus, it follows that $u \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega))$.

In a second step, we differentiate (21) with respect to time to have the initial and boundary value problem

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = \frac{\partial H}{\partial t}, \tag{52}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Gamma, \tag{53}$$

$$\frac{\partial u}{\partial t}(0) = \Delta u_0 - u_0 - f(u_0) + H_0. \tag{54}$$

Note that, if $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $H_0 \in L^2(\Omega)$, then $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ and

$$\left\|\frac{\partial u}{\partial t}(0)\right\| \leqslant Q(\|u_0\|_{H^2(\Omega)}, \|H_0\|). \tag{55}$$

Indeed, it follows from the continuity of f and the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\bar\Omega)$ that $\|f(u_0)\| \leqslant Q(\|u_0\|_{H^2(\Omega)})$.

Multiplying (52) by $\frac{\partial u}{\partial t}$, we obtain, in view of (28),

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|^2 + c\left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)} \leqslant c'\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2\right),\ c > 0. \tag{56}$$

Summing then (49) and δ_5 times (56), where $\delta_5 > 0$ is chosen small enough, we find a differential inequality of the form

$$\frac{dE_4}{dt} + c\left(E_4 + \|u\|^2_{H^2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)}\right) \leqslant c',\ c > 0, \tag{57}$$

where

$$E_4 = E_3 + \delta_5\left\|\frac{\partial u}{\partial t}\right\|^2 \tag{58}$$

satisfies

$$E_4 \geqslant c\bigg(\|u\|^2_{H^1(\Omega)}+\int_\Omega F(u)\,dx+\left\|\frac{\partial u}{\partial t}\right\|^2+\|H\|^2_{H^1(\Omega)}+\left\|\frac{\partial H}{\partial t}\right\|^2_{H^1(\Omega)}\bigg)-c',\ c>0, \tag{59}$$

which gives $\frac{\partial u}{\partial t} \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega))$.

We finally rewrite (21) as an elliptic equation, for $t>0$ fixed,

$$-\Delta u + u + f(u) = -\frac{\partial u}{\partial t} + H,\ u=0 \quad \text{on} \quad \Gamma. \tag{60}$$

Multiplying (60) by $-\Delta u$, we have, owing to (28),

$$\|\Delta u\|^2 \leqslant c\bigg(\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|H\|^2\bigg),$$

hence, owing to classical regularity results,

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant cE_4(t) + c',\ t \geqslant 0, \tag{61}$$

so that $u \in L^\infty(0,T,H^2(\Omega))$.

Having this, we multiply (22) by $-\Delta\frac{\partial H}{\partial t}$ and $-\Delta H$ to obtain

$$\frac{d}{dt}\bigg(\|\Delta H\|^2 + \left\|\nabla\frac{\partial H}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial H}{\partial t}\right\|^2\bigg) + \left\|\nabla\frac{\partial H}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial H}{\partial t}\right\|^2 \leqslant \|\Delta u\|^2 \tag{62}$$

and

$$\begin{aligned}\frac{d}{dt}\bigg(\|\nabla H\|^2 + \|\Delta H\|^2 + 2\Big(\Big(\nabla\frac{\partial H}{\partial t},\nabla H\Big)\Big) + 2\Big(\Big(\Delta\frac{\partial H}{\partial t},\Delta H\Big)\Big)\bigg) + \|\Delta H\|^2 \\ \leqslant \|\Delta u\|^2 + 2\bigg(\left\|\nabla\frac{\partial H}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial H}{\partial t}\right\|^2\bigg),\end{aligned} \tag{63}$$

respectively. Summing (62) and δ_6 times (63), where $\delta_6>0$ is chosen small enough, we find, in view of (61), a differential inequality of the form

$$\frac{dE_5}{dt} + cE_5 \leqslant c'E_4 + c'',\ c>0, \tag{64}$$

where

$$\begin{aligned}E_5 = \|\Delta H\|^2 + \left\|\nabla\frac{\partial H}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial H}{\partial t}\right\|^2 \\ + \delta_6\bigg(\|\nabla H\|^2 + \|\Delta H\|^2 + s\Big(\Big(\nabla\frac{\partial H}{\partial t},\nabla H\Big)\Big) + 2\Big(\Big(\Delta\frac{\partial H}{\partial t},\Delta H\Big)\Big)\bigg)\end{aligned} \tag{65}$$

satisfies

$$E_5 \geqslant c\bigg(\|H\|^2_{H^2(\Omega)} + \left\|\frac{\partial H}{\partial t}\right\|^2_{H^2(\Omega)}\bigg),\ c>0. \tag{66}$$

Gronwall's lemma then yields that $H, \frac{\partial H}{\partial t} \in L^\infty(0,T,H^2(\Omega))$.

In particular, it follows from (57) and Gronwall's lemma that

$$E_4(t) \leqslant e^{-ct}E_4(0) + c',\ c>0,\ t\geqslant 0, \tag{67}$$

which yields, owing to (59), the continuity of f and the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\bar{\Omega})$,

$$\begin{aligned} &\|u(t)\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial t}(t)\right\|^2 + \|H(t)\|^2_{H^1(\Omega)} + \left\|\frac{\partial H}{\partial t}\right\|^2_{H^1(\Omega)} \\ &\quad \leqslant e^{-ct} Q(\|u_0\|^2_{H^2(\Omega)}, \|H_0\|^2_{H^1(\Omega)}, \|H_1\|^2_{H^1(\Omega)}) + c',\ c > 0,\ t \geqslant 0. \end{aligned} \tag{68}$$

It then follows from (61), (67) and (68) that

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant e^{-ct} Q(\|u_0\|^2_{H^2(\Omega)}, \|H_0\|^2_{H^1(\Omega)}, \|H_1\|^2_{H^1(\Omega)}) + c',\ c > 0,\ t \geqslant 0, \tag{69}$$

and from (64), (66)-(68) and Gronwall's lemma that

$$\begin{aligned} &\|H(t)\|^2_{H^2(\Omega)} + \left\|\frac{\partial H}{\partial t}\right\|^2_{H^2(\Omega)} \\ &\quad \leqslant e^{-ct} Q(\|u_0\|^2_{H^2(\Omega)}, \|H_0\|^2_{H^2(\Omega)}, \|H_1\|^2_{H^2(\Omega)}) + c',\ c > 0,\ t \geqslant 0. \end{aligned} \tag{70}$$

3. **Existence and uniqueness of solutions.** One of the difficulties here is precisely to ensure that the order parameter u remains in the physical interval $(-1, +1)$, in order to give a meaning to the equations. We should note that the values -1 and $+1$ correspond to the pure phases. To prove the well-posedness of our problem, it suffices to obtain an estimate of H in $L^\infty((0,T) \times \Omega)$ (see [27]). We start with the following result.

Lemma 3.1. *Assume that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and that $\|u_0\|_{L^\infty} < 1$. Then, the order parameter u satisfies the strict separation property*

$$\|u(t)\|_{L^\infty} \leqslant \delta,\ t \in [0,T],\ \forall T > 0,$$

for some $\delta \in (0,1)$ depending on T.

Proof. It follows from the previous section that $H \in L^\infty(0,T,H^2(\Omega))$ and, since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we see that $H \in L^\infty((0,T) \times \Omega)$.

We set $v = u - \delta$, where $\delta \in (0,1)$. We have

$$\frac{\partial v}{\partial t} - \Delta v + v + f(u) - f(\delta) = H - f(\delta) - \delta. \tag{71}$$

Set now $v^+ = \max\{0, v\}$. Multiplying (71) by v^+, we find, integrating over Ω,

$$\frac{1}{2}\frac{d}{dt}\|v^+\|^2 + \|v^+\|^2_{H^1(\Omega)} + \int_\Omega (f(u) - f(\delta))v^+\,dx = ((H - f(\delta) - \delta, v^+)). \tag{72}$$

We note that, by definition, $v^+ = 0$ on Γ, since $v = 0$ on Γ. Then, thanks to (28),

$$\int_\Omega (f(u) - f(\delta))v^+\,dx \geqslant -c_1\|v^+\|^2, \tag{73}$$

whence

$$\frac{1}{2}\frac{d}{dt}\|v^+\|^2 + \|v^+\|^2_{H^1(\Omega)} \leqslant c_1\|v^+\|^2 + ((H - f(\delta) - \delta, v^+)). \tag{74}$$

By choosing δ such that

$$f(\delta) + \delta \geqslant \|H\|_{L^\infty} \quad \text{and} \quad \|u_0\|_{L^\infty} \leq \delta, \tag{75}$$

we then deduce that

$$\frac{d}{dt}\|v^+\|^2 + \|v^+\|^2_{H^1(\Omega)} \leqslant 2c_1\|v^+\|^2. \tag{76}$$

In particular,

$$\frac{d}{dt}\|v^+\|^2 \leqslant 2c_1\|v^+\|^2. \tag{77}$$

Gronwall's lemma then yields, noting that $v^+(0) = 0$,

$$\|v^+(t)\|^2 \leqslant 0. \tag{78}$$

This means that

$$v^+(t) = 0, \ \forall t \geqslant 0, \tag{79}$$

and, as $v \leqslant v^+$, then

$$v(t,x) \leqslant 0, \ \forall t \geqslant 0, \quad \text{a.e. } x \in \Omega \ (t \in [0,T]).$$

Therefore,

$$u(t,x) \leqslant \delta, \ \forall t \geqslant 0, \quad \text{a.e. } x \in \Omega. \tag{80}$$

As f is an odd function, we set $v = u - \lambda$, with $\lambda = -\delta$. We define the quantity $v_- = \min\{0, v\}$. Proceeding as above, replacing δ by λ, we obtain

$$\|v_-(t)\|^2 \leqslant 0, \ \text{since } v_-(0) = 0. \tag{81}$$

Consequently,

$$v_-(t) = 0, \ \forall t \geqslant 0.$$

Since $v \geqslant v_-$, there holds

$$v(t,x) \geqslant 0, \forall t \geqslant 0 \ \text{ a.e. } x \in \Omega,$$

This means that

$$u(t,x) \geqslant \lambda,$$

which is equivalent to

$$u(t,x) \geqslant -\delta, \ \forall t \geqslant 0 \ \text{ a.e. } x \in \Omega. \tag{82}$$

Finally,

$$\|u\|_{L^\infty((0,T)\times\Omega)} \leqslant \delta < 1. \tag{83}$$

Therefore, the order parameter u is strictly separated from the singular points of f. □

Theorem 3.1. *Let $(u_0, H_0, H_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ be such that $\|u_0\|_{L^\infty(\Omega)} < 1$. Then the problem (21)-(24) admits a unique solution $\left(u, H, \frac{\partial H}{\partial t}\right)$ such that*

$$\left(u, H, \frac{\partial H}{\partial t}\right) \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega))^3$$

and

$$\frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

Furtheremore, there exists a constant $\delta = \delta(T, u_0) \in (0,1)$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant \delta, \ \forall t \in [0,T], \forall T > 0.$$

Proof. **a) Existence:** We first regularize the function f by a $\mathcal{C}^1$ function f_δ defined by

$$f_\delta(s) = \begin{cases} f(-\delta) + f'(-\delta)(s+\delta), & \text{if} \quad s \in (-\infty, -\delta], \\ f(s), & \text{if} \quad s \in [-\delta, \delta], \\ f(\delta) + f'(\delta)(s-\delta), & \text{if} \quad s \in [\delta, +\infty), \end{cases}$$

where δ is the constant defined above. We can choose δ sufficiently close to 1 so that

$$f(\delta) \geqslant 0 \quad \text{and} \quad f'(\delta) \geqslant 0,$$

taking δ small enough if necessary.

We then consider the problem (21)-(24) with f replaced by f_δ and u replaced by u^δ, that is,

$$(\mathcal{P}_\delta) : \begin{cases} \dfrac{\partial u^\delta}{\partial t} - \Delta u^\delta + u^\delta + f_\delta(u^\delta) = H, \\ (I-\Delta)\left(\dfrac{\partial^2 H}{\partial t^2} + \dfrac{\partial H}{\partial t}\right) - \Delta H = -\Delta u^\delta, \\ u^\delta = H = 0, \quad \text{on} \quad \Gamma, \\ u^\delta|_{t=0} = u_0^\delta, \ H|_{t=0} = H_0, \ \dfrac{\partial H}{\partial t}|_{t=0} = H_1. \end{cases}$$

It is known that problem $(\mathcal{P}_\delta)$ admits a unique solution (see [11]). We further have

Lemma 3.2. *We assume that*

$$F_\delta = \int_0^s f_\delta(\tau)\, d\tau.$$

The functions f_δ and F_δ satisfy the following properties:

$$f'_\delta(s) \geqslant -c_1 \quad \text{and} \quad -c_0 \leqslant F_\delta(s) \quad \forall s \in \mathbb{R},$$

where c_0 and c_1 are the positive constants in (27) and (28) (taking δ smaller if necessary).

Proof. We consider, e.g., the case where $s \in]\delta, +\infty)$ and we have

$$f_\delta(s) = f'(\delta)(s-\delta) + f(\delta).$$

It is clear that

$$f'_\delta(s) = f'(\delta) \geqslant -c_1, \quad \forall s \in]\delta, +\infty).$$

Furthermore,

$$\begin{aligned} F_\delta(s) &= \int_0^s f_\delta(\tau)\, d\tau \\ &= \int_0^\delta f_\delta(\tau)\, d\tau + \int_\delta^s f_\delta(\tau)\, d\tau \\ &= \int_0^\delta f(\tau)\, d\tau + \int_\delta^s f_\delta(\tau)\, d\tau \\ &= F(\tau) + \int_\delta^s f_\delta(\tau)\, d\tau \\ &\geqslant -c_0 \quad (\text{since} \int_\delta^s f_\delta(\tau)\, d\tau \geqslant 0). \end{aligned}$$

□

As a consequence of Lemma 3.2, the *a priori* estimates established in Section 2 for the solutions to problem (21)-(24) still hold for the solutions to $(\mathcal{P}_\delta)$. In particular, we deduce from Lemma 3.1 that

$$\|u^\delta\|_{L^\infty(\Omega)} \leqslant \delta, \quad \forall t \geqslant 0.$$

Hence we have $f_\delta(u^\delta) = f(u^\delta)$ and we conclude that $(u^\delta, H^\delta, \frac{\partial H^\delta}{\partial t})$ is also a solution to (21)-(24).

b) Uniqueness: Let $\left(u^{(1)}, H^{(1)}, \frac{\partial H^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, H^{(2)}, \frac{\partial H^{(2)}}{\partial t}\right)$ be two solutions to (21)-(24) with initial data $\left(u_0^{(1)}, H_0^{(1)}, H_1^{(1)}\right)$ and $\left(u_0^{(2)}, H_0^{(2)}, H_1^{(2)}\right)$, respectively. We set

$$\left(u, H, \frac{\partial H}{\partial t}\right) = \left(u^{(1)}, H^{(1)}, \frac{\partial H^{(1)}}{\partial t}\right) - \left(u^{(2)}, H^{(2)}, \frac{\partial H^{(2)}}{\partial t}\right)$$

and

$$\left(u_0, H_0, H_1\right) = \left(u_0^{(1)}, H_0^{(1)}, H_1^{(1)}\right) - \left(u_0^{(2)}, H_0^{(2)}, H_1^{(2)}\right)$$

and have

$$\frac{\partial u}{\partial t} - \Delta u + u + f\big(u^{(1)}\big) - f\big(u^{(2)}\big) = H, \tag{84}$$

$$(I - \Delta)\left(\frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t}\right) - \Delta H = -\Delta u, \tag{85}$$

$$u = H = 0 \ \text{ on } \ \Gamma, \tag{86}$$

$$u|_{t=0} = u_0, \ H|_{t=0} = H_0, \ \frac{\partial H}{\partial t}|_{t=0} = H_1. \tag{87}$$

Multiplying (84) by u, we obtain, in view of (28),

$$\frac{d}{dt}\|u\|^2 + \|u\|^2_{H^1(\Omega)} \leqslant c(\|u\|^2 + \|H\|^2). \tag{88}$$

Multiplying then (85) by $(-\Delta)^{-1}\frac{\partial H}{\partial t}$, we find

$$\frac{d}{dt}\left(\|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2\right) + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2 \leqslant \|u\|^2. \tag{89}$$

Summing finally (88) and (89), we have a differential inequality of the form

$$\frac{dE_6}{dt} \leqslant cE_6, \tag{90}$$

where

$$E_6 = \|u\|^2 + \|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2_{-1} + \left\|\frac{\partial H}{\partial t}\right\|^2 \tag{91}$$

satisfies

$$E_6 \geqslant c\left(\|u\|^2 + \|H\|^2 + \left\|\frac{\partial H}{\partial t}\right\|^2\right), \ c > 0. \tag{92}$$

It thus follows from (90)-(92) and Gronwall's lemma that

$$\|u(t)\|^2 + \|H(t)\|^2 + \left\|\frac{\partial H}{\partial t}(t)\right\|^2 \leqslant ce^{c't}(\|u_0\|^2 + \|H_0\|^2 + \|H_1\|^2), \ t \geqslant 0, \tag{93}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the $L^2 \times L^2 \times L^2$-topology. □

It follows from Theorem 3.1 that we can define the family of solving operators

$$S(t) : \Phi \to \Phi,\ (u_0, H_0, H_1) \mapsto \Big(u(t), H(t), \frac{\partial H}{\partial t}(t)\Big),\ t \geqslant 0,$$

where

$$\Phi = \{(u, H, \frac{\partial H}{\partial t}) \in (H^2(\Omega) \cap H_0^1(\Omega))^3;\ \|u\|_{L^\infty} < 1\}.$$

Furthermore, this family of solving operators forms a semigroup, that is, $S(0) = I$ and $S(t + \tau) = S(t) \circ S(\tau)$, $\forall t, \tau \geqslant 0$, which is continuous with respect to the L^2-topology.

4. **Dissipativity.** In this section, we study the dissipativity of our system. Moreover, one difficulty is that δ depends on the initial data and time T; note that the constant δ that appears in the strict separation property satisfied by the order parameter u is such that $\|u_0\|_{L^\infty(\Omega)} \leqslant \delta < 1$. Our aim is therefore to have an estimate that does not depend on the initial data nor on the time, at least for large times. To do this, we proceed as in [27]; see also, e.g., [11], [12] and [19].

Let $R_0 > 0$ be given and assume that

$$\frac{1}{1 - \|u_0\|_{L^\infty(\Omega)}} + \|u_0\|^2_{H^2(\Omega)} + \|H_0\|^2_{H^2(\Omega)} + \|H_1\|^2_{H^2(\Omega)} \leqslant R_0^2.$$

We then have, owing to Theorem 3.1 and (69)-(70), the existence of $t_0 = t_0(R_0) \geqslant 0$ such that

$$\|H(t)\|_{L^\infty(\Omega)} \leqslant C, \quad \forall t \geqslant t_0, \tag{94}$$

where C is independent of R_0. Furthermore, there holds

$$\|H(t)\|_{L^\infty(\Omega)} \leqslant \tilde{\delta}, \quad \forall t \geqslant 0, \tag{95}$$

where $\tilde{\delta} = \tilde{\delta}(R_0)$. Here, we can assume without loss of generality that $C \leqslant \tilde{\delta}$.

We now choose $\beta \in (0, 1)$ independent of R_0 and $t_1 \geqslant t_0$ such that

$$f(\beta) \geqslant C + 1 \tag{96}$$

and $\gamma(= \gamma(R_0)) = \dfrac{1 - \beta}{t_1}$ small enough so that

$$\gamma \leqslant 1, \quad f(1 - \gamma t_0) \geqslant \tilde{\delta} + 1. \tag{97}$$

We finally set

$$y_+(t) = \begin{cases} 1 - \gamma t, & \text{if} \quad 0 \leqslant t \leqslant t_1, \\ \beta, & \text{if} \quad t \geqslant t_1. \end{cases}$$

We have

$$\beta \leqslant y_+(t) < 1, \quad \forall t > 0,\ y_+(0) = 1. \tag{98}$$

Finally, we define the variable θ by

$$\theta = u - y_+. \tag{99}$$

We then have

$$\frac{\partial \theta}{\partial t} - \Delta\theta + \theta + f(u) - f(y_+) = G := H - f(y_+) - y_+'(t) - y_+(t),\ t > 0,\ t \neq t_1, \tag{100}$$

where y'_+ is the derivative of y_+. Furthermore, there holds, owing to (96),

$$G(t) \leqslant \begin{cases} \tilde{\delta} + 1 - f(1-\gamma t_0), & 0 < t \leqslant t_0, \\ C + 1 - f(\beta), \ t \geqslant t_0, & t \neq t_1, \end{cases}$$

hence, in view of (96) and (97),

$$G(t) \leqslant 0, \quad \forall t > 0, \quad t \neq t_1. \tag{101}$$

Setting

$$\theta^+ = \max\{\theta, 0\}, \tag{102}$$

we have, multipying (100) by θ^+ and integrating over Ω and by parts, in view of (101),

$$\frac{d}{dt}\|\theta^+\|^2 + \|\theta^+\|^2_{H^1(\Omega)} \leqslant c\|\theta^+\|^2, \quad t > 0, \quad t \neq t_1. \tag{103}$$

Using Gronwall's lemma and noting that θ^+ is continuous with respect to time and that $\theta^+(0) = 0$, we then deduce that

$$\theta^+(t) = 0, \quad \forall t \geqslant 0. \tag{104}$$

Therefore,

$$u(t) \leqslant y_+(t), \quad \forall t > 0, \tag{105}$$

and

$$u(t) \leqslant \beta, \quad \forall t \geqslant t_1. \tag{106}$$

Proceeding in a similar way to derive a lower bound, we finally deduce that there exists $\beta \in (0, 1)$ independent of R_0 such that

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant \beta, \quad \forall t \geqslant t_1, \ t_1 = t_1(R_0), \tag{107}$$

hence a dissipative L^∞-estimate on u.

The dynamical system $(S(t), \Phi)$ is thus dissipative (i.e., it possesses a bounded absorbing set $\mathcal{B}_0$, that is, $\forall B \in \Phi$ bounded, $\exists t_0 = t_0(B)$ such that $t \geqslant t_0$ implies $S(t)B \subset \mathcal{B}_0$; it is understood here that B bounded means that $\exists R \geqslant 0$ such that $\dfrac{1}{1 - \|u\|_{L^\infty(\Omega)}} + \|u\|^2_{H^2(\Omega)} + \|H\|^2_{H^2(\Omega)} + \|\dfrac{\partial H}{\partial t}\|^2_{H^2(\Omega)} \leqslant R^2, \ \forall (u, H, \dfrac{\partial H}{\partial t}) \in B$).

Theorem 4.1. *The semigroup $S(t), t \geqslant 0$, associated to our system is dissipative on Φ, i.e., it possesses a bounded absorbing set $\mathcal{B}_0$ in Φ.*

5. **Existence of the global attractor.**

Theorem 5.1. *Under the hypotheses of Theorem 4.1, the semigroup $S(t), \ t \geqslant 0$, defined from $\mathcal{B}_0$ into itself possesses the connected global attractor denoted by $\mathcal{A}$.*

Proof. According to the previous section, it is known that the semigroup possesses a bounded absorbing set $\mathcal{B}_0$ in Φ. To prove the existence of the global attractor $\mathcal{A}$, it suffices to prove that the semigroup is asymptotically compact in the sense of the Kuratowski measure of noncompactness.

We consider the following decomposition:

$$\Big(u, H, \frac{\partial H}{\partial t}\Big) = \Big(v, a, \frac{\partial a}{\partial t}\Big) + \Big(w, b, \frac{\partial b}{\partial t}\Big),$$

where $(v, a, \dfrac{\partial a}{\partial t})$ is solution of

$$\frac{\partial v}{\partial t} - \Delta v + v = a, \tag{108}$$

$$(I-\Delta)\left(\frac{\partial^2 a}{\partial t^2}+\frac{\partial a}{\partial t}\right)-\Delta a=-\Delta v, \tag{109}$$

$$v=a=0, \quad \text{on} \quad \Gamma, \tag{110}$$

$$v(0)=u_0,\ a(0)=H_0,\ \frac{\partial a}{\partial t}(0)=H_1 \tag{111}$$

and $\left(w,b,\dfrac{\partial b}{\partial t}\right)$ is solution of

$$\frac{\partial w}{\partial t}-\Delta w+w+f(u)=b, \tag{112}$$

$$(I-\Delta)\left(\frac{\partial^2 b}{\partial t^2}+\frac{\partial b}{\partial t}\right)-\Delta b=-\Delta w, \tag{113}$$

$$w=b=0, \quad \text{on} \quad \Gamma, \tag{114}$$

$$v(0)=a(0)=\frac{\partial a}{\partial t}(0)=0, \tag{115}$$

with the initial data in the bounded absorbing set $\mathcal{B}_0$. We will now write a certain number of *a priori* estimates. First, repeating the same estimates leading to (69)-(70), but now taking $f\equiv 0$, we obtain

$$\begin{aligned}\|v(t)\|^2_{H^2(\Omega)}+\|a(t)\|^2_{H^2(\Omega)}+\left\|\frac{\partial a}{\partial t}\right\|^2_{H^2(\Omega)}&\\ \leqslant e^{-ct}(\|u_0\|^2_{H^2(\Omega)}+\|H_0\|^2_{H^2(\Omega)}+\|H_1\|^2_{H^2(\Omega)}),\ \forall t\geqslant 0.&\end{aligned} \tag{116}$$

We can see that $S_1(t)(u_0,H_0,H_1)=\left(v(t),a(t),\dfrac{\partial a}{\partial t}\right)$ tends to zero as t tends to infinity.

We now consider the system (112)-(115). Multiplying (112) by $\Delta^2 w+\Delta^2\dfrac{\partial w}{\partial t}$, integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\nabla w\|^2+\|\Delta w\|^2+\|\nabla\Delta w\|^2\right]+\|\Delta w\|^2+\|\nabla\Delta w\|^2+\left\|\Delta\frac{\partial w}{\partial t}\right\|^2\\ &\quad=((\Delta b,\Delta w))+\left(\left(\Delta b,\Delta\frac{\partial w}{\partial t}\right)\right)-\left(\left(\Delta f(u),\Delta\frac{\partial w}{\partial t}\right)\right)-((\Delta f(u),\Delta w)).\end{aligned} \tag{117}$$

Multiplying now (113) by $\Delta^2 b+\Delta^2\dfrac{\partial b}{\partial t}$ and integrate over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\Delta b\|^2+\|\nabla\Delta b\|^2+\|\nabla b\|^2+\left\|\Delta\frac{\partial b}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2+2\left(\left(\nabla b,\nabla\frac{\partial b}{\partial t}\right)\right)\right.\\ &\quad\left.+2\left(\left(\Delta b,\Delta\frac{\partial b}{\partial t}\right)\right)\right]+\|\nabla\Delta b\|^2+\left\|\Delta\frac{\partial b}{\partial t}\right\|^2+\left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2\\ &\quad=\left(\left(\nabla\Delta w,\nabla\Delta\frac{\partial b}{\partial t}\right)\right)+\left(\left(\nabla\Delta w,\nabla\Delta b\right)\right).\end{aligned} \tag{118}$$

From Hölder's inequality, we write

$$\left|\left(\left(\Delta f(u),\Delta\frac{\partial w}{\partial t}\right)\right)\right|\leqslant\frac{1}{2\epsilon}\|\Delta f(u)\|^2+\frac{\epsilon}{2}\left\|\Delta\frac{\partial w}{\partial t}\right\|^2 \tag{119}$$

and

$$\left|\left(\left(\Delta f(u), \Delta w\right)\right)\right| \leqslant \frac{1}{2\epsilon}\|\Delta f(u)\|^2 + \frac{\epsilon}{2}\|\Delta w\|^2 \leqslant \frac{1}{2\epsilon}\|\Delta f(u)\|^2 + c\epsilon\|\nabla\Delta w\|^2. \quad (120)$$

Summing (117) and (118), insering (119) and (120) in the resulting estimate and choosing $\epsilon > 0$ small enough so that $2 - c\epsilon > 0$, we obtain

$$\frac{dE}{dt} + c\left[\|\Delta w\|^2 + \|\nabla\Delta w\|^2 + \left\|\Delta\frac{\partial w}{\partial t}\right\|^2 + \|\nabla\Delta b\|^2 + \left\|\Delta\frac{\partial b}{\partial t}\right\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2\right] \leqslant c'\|\Delta f(u)\|^2, \quad (121)$$

where E satisfies

$$E \geqslant c\left(\|\nabla\Delta w\|^2 + \|\nabla\Delta b\|^2 + \|\Delta w\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2\right) - c',\ c > 0. \quad (122)$$

Integrating (121) over $(0, t)$ and using (115) and (122), we get

$$\|\nabla\Delta w(t)\|^2 + \|\Delta w(t)\|^2 + \|\nabla\Delta b(t)\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2 \leqslant c'\int_0^t \|\Delta f(u)\|^2\, ds. \quad (123)$$

By (69), we have

$$\int_0^t \|\Delta f(u)\|^2\, ds \leqslant C_{T,\|(u_0,H_0,H_1)\|_{(H^2(\Omega))^3},\mathcal{B}_0}. \quad (124)$$

Finally inserting (124) into (123), we have

$$\|w(t)\|^2_{H^3(\Omega)} + \|b(t)\|^2_{H^3(\Omega)} + \left\|\frac{\partial b}{\partial t}\right\|^2_{H^3(\Omega)} \leqslant C_{T,\|(u_0,H_0,H_1)\|_{(H^2(\Omega))^3},\mathcal{B}_0}. \quad (125)$$

Hence, the operator $S_2(t)(u_0, H_0, H_1) = \left(w(t), b(t), \frac{\partial b}{\partial t}(t)\right)$ is asymptotically compact in the sense of the Kuratowski measure of noncompactness, which proves the existence of the global attractor $\mathcal{A}$. □

6. Discretization of the nonconserved Caginalp phase-field system.

In this section, we present the spatial discretization using a finite element method with $\mathbb{P}_1$ continuous piecewise linear functions and a first-order semi-implicit scheme for the time marching scheme.

6.1. Spatial discretization.

We let Ω be a convex, planar domain and $\mathbf{T}_h$ be a regular, quasi-uniform triangulation of Ω with triangles of maximum size $h < 1$. Setting $V_h = \{v_h \in C^0(\bar{\Omega}); v_h|_{\mathbf{T}_h} \in \mathbb{P}_1(\mathbf{T}_h),\ \forall T \in \mathbf{T}_h\}$ a finite-dimensional subspace of $H^1(\Omega)$, where $\mathbb{P}_1$ is the set of all polynomials of degree $\leqslant 1$ with real coefficients, we consider the weak formulation of (21)-(22):

Find $u_h, H_h \in V_h$ such that, $\forall \phi_h \in V_h$,

$$\begin{aligned}
\left(\left(\frac{\partial u_h}{\partial t} - \varepsilon\Delta u_h + u_h + \frac{1}{\varepsilon}f(u_h), \phi_h\right)\right) &= \left(\left(H_h, \phi_h\right)\right), \\
\left(\left((I_d - \Delta)\left(\tau\frac{\partial^2 H_h}{\partial t^2} + \frac{\partial H_h}{\partial t}\right) - \Delta H_h, \phi_h\right)\right) &= -\left(\left(\Delta u_h, \phi_h\right)\right), \\
u_h|_{t=0} = f_{u_h(x,y,0)}, \quad H_h|_{t=0} = f_{H_h(x,y,0)}, \quad \frac{\partial H_h}{\partial t}|_{t=0} &= g_{H_h(x,y,0)}.
\end{aligned} \quad (126)$$

6.2. **Time marching scheme.** We will discretize system (126) in time using a first-order semi-implicit scheme. To this end, let us denote by (u_h^{n+1}, H_h^{n+1}) and (u_h^n, H_h^n) the approximate values at time $t = t^{n+1}$ and $t = t^n$ respectively and by δt the time step. Then, owing to (126), the unknown fields at time $t = t^{n+1}$ are defined as the solution of:

$$\left(\left((I_d + \delta t \cdot (I_d - \varepsilon\Delta))\, u_h^{n+1} - \delta t \cdot H_h^{n+1}, \phi_h \right)\right) = \left(\left(u_h^n - \delta t \frac{1}{\varepsilon} \cdot f(u_h^{n+1}), \phi_h \right)\right),$$

$$\begin{aligned} &\left(\left(\left((\tau + \delta t)\,(I_d - \Delta) - \delta t^2 \cdot \Delta\right) H_h^{n+1} + \delta t^2 \Delta u_h^{n+1}, \phi_h \right)\right) \\ &\quad = \left(\left((2\tau + \delta t) \cdot (I_d - \Delta) H_h^n - \tau\,(I_d - \Delta)\, H_h^{n-1}, \phi_h \right)\right) \end{aligned} \tag{127}$$

$$u_h^n|_{t=0} = f_{u_h(x,y,0)},\ H_h^{n-1}|_{t=0} = f_{H_h(x,y,0)}, H_h^n|_{t=0} = H_h^{n-1}|_{t=0} + \delta t \cdot g_{H_h(x,y,0)},$$

in which (127) can be written equivalently in the following matrix form ($\mathbf{AX} = \mathbf{B}$):

$$\begin{pmatrix} (I_d + \delta t \cdot (I_d - \varepsilon\Delta))\,(\cdot) & \delta t I_d(\cdot) \\ \delta t^2 \Delta(\cdot) & \left((\tau + \delta t)\,(I_d - \Delta) - \delta t^2 \cdot \Delta\right)(\cdot) \end{pmatrix} \begin{pmatrix} u_h^{n+1} \\ H_h^{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{F}(u_h^n, u_h^{n+1}) \\ \mathbf{G}(H_h^n, H_h^{n-1}) \end{pmatrix}, \tag{128}$$

where

$$\begin{aligned} \mathbf{F}(u_h^n, u_h^{n+1}) &= u_h^n - \delta t \frac{1}{\varepsilon} \cdot f(u_h^{n+1}), \\ \mathbf{G}(H_h^n, H_h^{n-1}) &= (2\tau + \delta t) \cdot (I_d - \Delta)\, H_h^n - \tau\,(I_d - \Delta)\, H_h^{n-1}. \end{aligned} \tag{129}$$

Finally, the simplest method to solve (128)–(129) is to use Picard's iterate as follows:

Algorithm 1:

```
Set     u_h^n = f^0_{u_h}
Set     H_h^{n-1} = f^0_{H_h}, H_h^n = H_h^{n-1} + δt · g^0_{H_h}
ComputeA(if not using adaptmesh)
For t = 2 · δt : δt : T
        Compute A(if using adaptmesh)
        Compute G(H_h^n, H_h^{n-1})
        Set u_hi^n = u_h^n, err = 1, (for the fixed point method used for f(u))
        while err ⩾ 1e^{-10}
             Compute F(u_h^n, u_hi^n)
             Set X = [u_h^{n+1}, H_h^{n+1}],      Compute B,     Solve AX = B
             Compute err = norm (u_hi^n − u_h^{n+1})/norm (u_h^{n+1})
             Actualize u_hi^n = u_h^{n+1}
        end while
    Setu_h^n = u_h^{n+1}, H_h^{n-1} = H_h^n, H_h^n = H_h^{n+1}
End for
```

7. **Numerical simulations.** We perform several numerical simulations using the FreeFem++ software [24], comparing the nonconserved Caginalp phase-field system (128)-(129) with the cubic nonlinear term $f(s)$ satisfying (19) (respectively, (20)) and the logarithmic one $f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\frac{1+s}{1-s}\right)$ when $(\kappa_0, \kappa_1) = (\ln(3), 1)$ (respectively, $(\kappa_0, \kappa_1) = (\ln(6), 1)$). We first start by considering the rate of convergence of the first-order in time semi-implicit scheme. We then compute the propagation of a cross function for u and a constant enthalpy H.

7.1. **Rate of convergence.** In this subsection, we check the convergence rates of the nonconserved Caginalp phase-field system (128)-(129), where the values of the L_2, H_1 error estimates for u and H and their corresponding convergence rates are given in Tables 1 $\longrightarrow$ 6.

We first start by considering the rate of convergence of the first-order semi-implicit scheme in time, where we use $\mathbb{P}_1$ continuous piecewise linear functions for the finite element space for u and H, periodic boundary conditions for u and H and as exact solution on the unit square $[0, L] \times [0, L]$, $L = 1$, the functions

$$u_{ex} = .1 \sin\left(\frac{2\,\pi\, x}{L} - t\right) \cos\left(\frac{2\,\pi\, y}{L} - t\right)$$

and

$$H_{ex} = .1 \cos\left(\frac{2\,\pi\, x}{L} - t\right) \cos\left(\frac{2\,\pi\, y}{L} - t\right),$$

adding an appropriate right-hand side function. We take $\kappa_0 = \ln(3)$, $\kappa_1 = 1$, $\varepsilon \in \{.1, .01, .001\}$, $\tau = 3.e-2$, $\Delta t = \varepsilon \frac{L}{N^2}$ with $N \in \{10, 20, 40, 80, 160\}$ and we measure at time $T = \varepsilon$ the following errors: $N_{L^2}(u) = \|u_h - u_{ex}\|_{L^2}$, $N_{H^1}(u) = \|u_h - u_{ex}\|_{H^1}$, $N_{L^2}(H) = \|H_h - H_{ex}\|_{L^2}$, $N_{H^1}(H) = \|H_h - H_{ex}\|_{H^1}$.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:02	0.00147	-	0.00124	-	0.0573	-	0.05544	-
1/4	00:00:22	0.00038	0.98	0.0003	0.98	0.02927	0.48	0.02828	0.49
1/16	00:05:58	9.5e-05	0.99	8.1e-05	0.99	0.01472	0.49	0.01422	0.49
1/64	01:50:22	2.4e-05	0.99	2e-05	0.99	0.00737	0.49	0.00711	0.49
1/256	22:06:19	6e-06	1	5e-06	0.99	0.00369	0.5	0.00357	0.49

TABLE 1. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .1$, $\tau = .03$ and $f(s) = .83(s^3 - .5^2 s)$.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:02	0.00146	-	0.00124	-	0.05725	-	0.05544	-
1/4	00:00:25	0.00038	0.98	0.0003	0.98	0.02927	0.48	0.02828	0.49
1/16	00:07:44	9.5e-05	0.99	8.1e-05	0.99	0.01472	0.49	0.01422	0.49
1/64	01:52:09	2.4e-05	0.99	2e-05	0.99	0.00737	0.49	0.00711	0.49
1/256	23:06:55	6e-06	1	5e-06	0.99	0.00369	0.5	0.00357	0.49

TABLE 2. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .1$, $\tau = .03$ and $f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\frac{1+s}{1-s}\right)$.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:01	0.00042	-	0.00039	-	0.02458	-	0.02397	-
1/4	00:00:22	0.00011	0.98	0.0001	0.98	0.01249	0.49	0.01215	0.49
1/16	00:07:09	2.7e-05	0.99	2.5e-05	0.99	0.00627	0.49	0.00611	0.49
1/64	01:27:20	7e-06	0.99	6e-06	0.99	0.00314	0.49	0.00306	0.49
1/256	22:16:47	2e-06	1	2e-06	0.99	0.00157	0.5	0.00154	0.49

TABLE 3. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .01$, $\tau = .03$ and $f(s) = .83(s^3 - .5^2 s)$.

We can note that in both cases, for $\varepsilon \in \{.1, .01, .001\}$, we obtain an optimal convergence rate in time of order 1 for the $L^2(\Omega\times]0,T[)$ norm for u and H and of order .5 for the $L^2(0,T;H^1(\Omega)^2)$ norm for u and H, which confirms the convergence of the first-order semi-implicit scheme in time for the nonconserved Caginalp phase-field system. We can note that we would obtain the same results for the second set of nonlinear terms mentioned in the introduction.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:01	0.00042	-	0.00039	-	0.02457	-	0.02397	-
1/4	00:00:24	0.00011	0.98	0.0001	0.98	0.01248	0.49	0.01215	0.49
1/16	00:07:41	2.7e-05	0.99	2.5e-05	0.99	0.00627	0.49	0.00611	0.49
1/64	01:26:11	7e-06	0.99	6e-06	0.99	0.00314	0.49	0.00306	0.49
1/256	23:06:30	2e-06	1	2e-06	0.99	0.00157	0.5	0.00154	0.49

TABLE 4. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .01$, $\tau = .03$ and $f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\dfrac{1+s}{1-s}\right)$.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:01	0.00013	-	0.00012	-	0.01227	-	0.01186	-
1/4	00:00:22	3.4e-05	0.98	3.2e-05	0.98	0.00625	0.49	0.00603	0.49
1/16	00:07:19	9e-06	0.99	8e-06	0.99	0.00314	0.49	0.00303	0.49
1/64	01:29:47	2e-06	0.99	2e-06	0.99	0.00157	0.49	0.00152	0.49
1/256	22:23:29	1e-06	1	1e-06	0.99	0.00079	0.5	0.00076	0.49

TABLE 5. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .001$, $\tau = .03$ and $f(s) = .83(s^3 - .5^2 s)$.

$10^2 \cdot \delta t$	CPU time	$N_{L^2}(u)$	rate	$N_{L^2}(H)$	rate	$N_{H^1}(u)$	rate	$N_{H^1}(H)$	rate
1/1	00:00:01	0.00013	-	0.00012	-	0.01226	-	0.011863	-
1/4	00:00:24	3.4e-05	0.98	3.2e-05	0.98	0.00624	0.49	0.00603	0.49
1/16	00:07:44	9e-06	0.99	8e-06	0.99	0.00314	0.49	0.00303	0.49
1/64	01:29:12	2e-06	0.99	2e-06	0.99	0.00157	0.49	0.00152	0.49
1/256	23:22:36	1e-06	1	1e-06	0.99	0.00079	0.5	0.00076	0.49

TABLE 6. L^2, H^1 norm and error for u and H for the nonconserved Caginalp phase-field system with $\varepsilon = .001$, $\tau = .03$ and $f(s) = -2\kappa_0 s + \kappa_1 \ln\left(\dfrac{1+s}{1-s}\right)$.

7.2. **Propagation in a square.** We present in this section the propagation of the solution in the square $[0,300] \times [0,300]$ of the cross function $f_u(x,y,0)$ defined in FreeFem++ as:

```
real amp=0.02,L=300.,x0=L/2.,y0=L/2.,R0=20.;
func fu=(amp*((x <= x0+R0) *(x >= x0-R0)) *(y <= y0+3*R0)*(y >=
    ➥ y0-3*R0)+amp*((x < x0-R0) *(x >= x0-3*R0)) *(y <= y0+R0)
    ➥*(y >= y0-R0)+amp*((x <= x0+3*R0) *(x > x0+R0)) *(y <= y0+
    ➥R0)*(y >= y0-R0))-amp/2.;
```

and we take $g_H(x,y,0) = 0$. We further take $\delta x = 2$, $\varepsilon \in \{.1, .01, .001\}$, $\delta t = \varepsilon \dfrac{L}{N^2}$, $\tau = 3.e-2$ and periodic boundary condition for u and H, taking into account that we choose different initial data for $f_H(x,y,0)$. Similar results can be obtained here, considering another initial cross function with `.02 < amp < 1.8`.

We also use here the `adaptmesh` of FreeFem++ with `uadapt=Hn+un,err=1.e-4` each 100 iteration, where, in that case, we obtain an error of order 10^{-5} with the solution without using adaptmesh as shown in [38] and we gain a lot of computational time.

We further note that the solution u starts from $[-.01, .01]$ and goes to one of the following values: $[-\alpha, \alpha]$, α, $-\alpha$ ($\alpha \in]-1., 1.[$); the solution can also explode depending on the value of ε and the initial datum for $f_H(x,y,0)$. This explains why we use periodic boundary conditions and not Dirichlet ones.

More precisely, we display in Tables 7 $\longrightarrow$ 9 the convergence of the solution u for different values of $f_H(x,y,0)$ when $f(s)$ satisfies (19), while in Tables 10 $\longrightarrow$ 12, we take $f(s)$ satisfying (20). We obtain opposite values of α when $f_H(x,y,0) = -\delta$ and $f_H(x,y,0) = \delta$, with $\delta > 0$. We can clearly see that, for (20), the logarithmic potential works much better than the polynomial one.

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	explose	$-.41$	$[-.37, .37]$	$.41$	$.44$	$.62$	$.94$	explose
pol	explose	$-.40$	$[-.37, .37]$	$.40$	$.43$	$.64$	explose	explose

TABLE 7. Comparison of the convergence of the solution u with $\varepsilon = .1$.

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	$-.76$	$[-.49, .49]$	$[-.49, .49]$	$[-.49, .49]$	$.49$	$.52$	$.67$	$.76$
pol	$-.86$	$[-.49, .49]$	$[-.49, .49]$	$[-.49, .49]$	$.49$	$.52$	$.70$	$.86$

TABLE 8. Comparison of the convergence of the solution u with $\varepsilon = .01$.

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	$-.56$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$.53$	$.56$
pol	$-.57$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$[-.50, .50]$	$.53$	$.57$

TABLE 9. Comparison of the convergence of the solution u with $\varepsilon = .001$.

We thus deduce that, the smaller ε is, the faster the solution converges to a constant (negative or positive) solution $0.5 < |\alpha| < 1$. We also note that, when

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	explose	explose	explose	explose	.92	.94	.99	explose
pol	explose	explose	explose	explose	explose	.94	explose	explose

TABLE 10. Comparison of the convergence of the solution u with $\varepsilon = .1$.

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	-.95	$[-.93, .93]$	$[-.93, .93]$	$[-.93, .93]$	$[-.93, .93]$	explose	.94	.95
pol	explose	explose	explose	explose	explose	explose	.96	explose

TABLE 11. Comparison of the convergence of the solution u with $\varepsilon = .01$.

f_H	-35	$-.1$	0	$.1$	$.2$	1.5	15	35
log	-.93	$[-.93, .93]$	$[-.93, .93]$	$[-.93, .93]$	$[-.93, .93]$	$[-.93, .93]$	explose	.93
pol	-.94	explose	explose	explose	explose	explose	explose	.94

TABLE 12. Comparison of the convergence of the solution u with $\varepsilon = .001$.

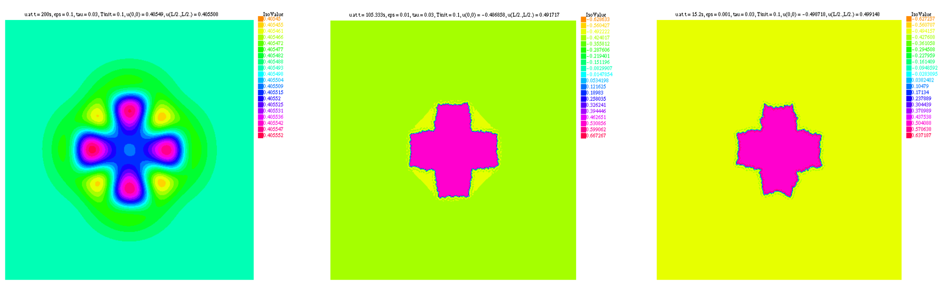

FIGURE 1. Solution u with $f_H = .1$ and logarithmic potential.

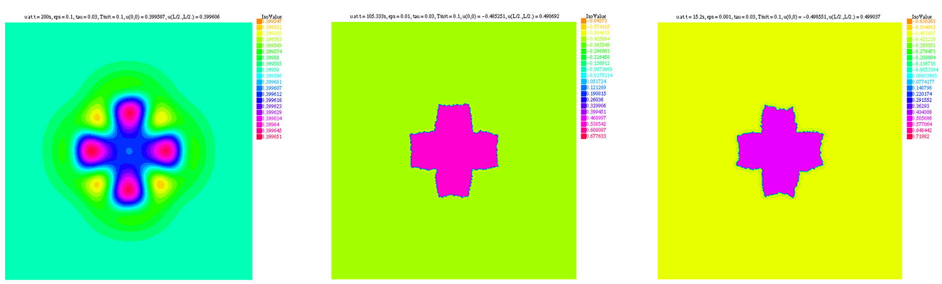

FIGURE 2. Solution u with $f_H = .1$ and cubic potential.

the solution lies between $[-0.5, 0.5]$, we need more iterations and time in order to possibly obtain the convergence.

In Figures 1 $\longrightarrow$ 8, we consider the convergence of u and H with different values of ε ($\varepsilon = 0.1$ (left), $\varepsilon = 0.01$ (center) and $\varepsilon = 0.001$ (right)) with $f_H(x, y, 0) = .1$ or $f_H(x, y, 0) = 1.5$ when $f(s)$ satisfies (19). We also note that we did not observe any influence of τ on the simulations.

Acknowledgments. The authors wish to thank Jean-Paul Chehab, Laurence Cherfils and Morgan Pierre for many stimulating discussions and useful comments.

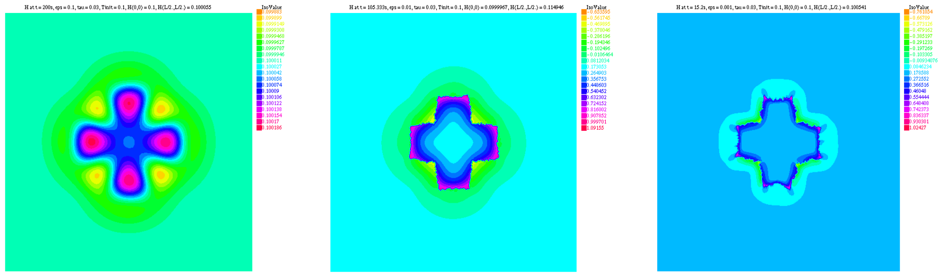

FIGURE 3. Solution H with $f_H = .1$ and logarithmic potential.

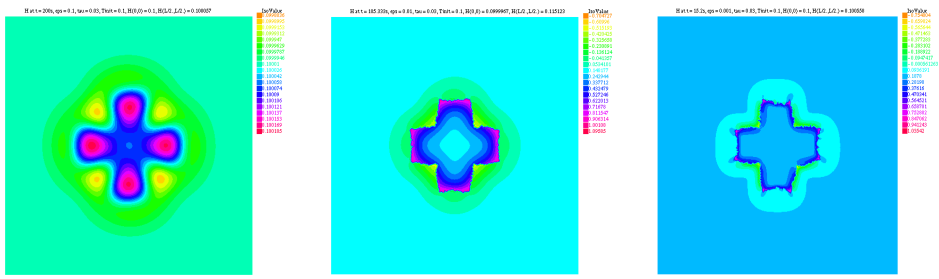

FIGURE 4. Solution H with $f_H = .1$ and cubic potential.

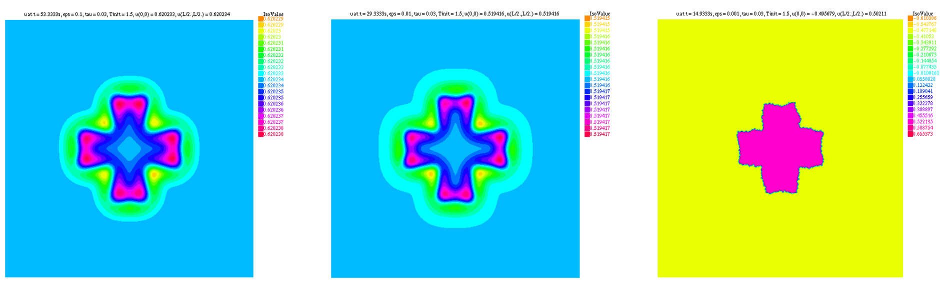

FIGURE 5. Solution u with $f_H = 1.5$ and logarithmic potential.

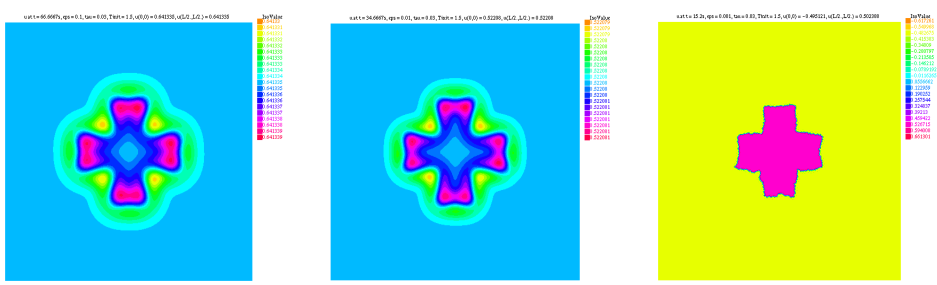

FIGURE 6. Solution u with $f_H = 1.5$ and cubic potential.

REFERENCES

[1] S. Aizicovici and E. Feireisl, Long-time stabilization of solutions to a phase-field model with memory, *J. Evol. Eqns.*, **1** (2001), 69–84.

[2] S. Aizicovici, E. Feireisl and F. Issard-Roch, Long-time convergence of solutions to a phase-field system, *Math. Methods Appl. Sci.*, **24** (2001), 277–287.

[3] D. Brochet, X. Chen and D. Hilhorst, Finite dimensional exponential attractors for the phase-field model, *Appl. Anal.*, **49** (1993), 197–212.

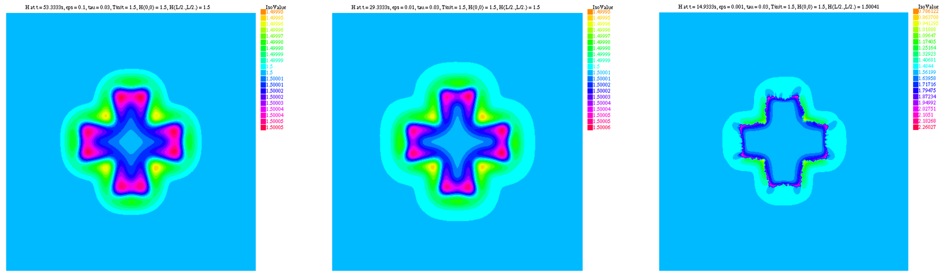

FIGURE 7. Solution H with $f_H = 1.5$ and logarithmic potential.

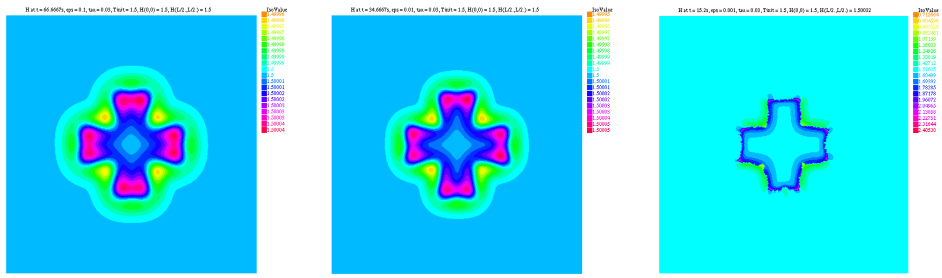

FIGURE 8. Solution H with $f_H = 1.5$ and cubic potential.

[4] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.

[5] G. Caginalp, An analysis of a phase-field model of a free boundary, *Arch. Rational Mech. Anal.,* **92** (1986), 205–245.

[6] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.,* **2** (1958), 258–267.

[7] P. J. Chen and M. E. Gurtin, On a theory of heat involving two temperatures, *J. Appl. Math. Phys. (ZAMP),* **19** (1968), 614–627.

[8] P. J. Chen, M. E. Gurtin and W. O. Williams, A note on a non-simple heat conduction, *J. Appl. Math. Phys. (ZAMP)*, **19** (1968), 969–970.

[9] P. J. Chen, M. E. Gurtin and W. O. Williams, On the thermodynamics of non-simple materials with two temperatures, *J. Appl. Math. Phys. (ZAMP)*, **20** (1969), 107–112.

[10] L. Cherfils and A. Miranville, Some results on the asymptotic behavior of the Caginalp system with singular potentials, *Adv. Math. Sci. Appl.,* **17** (2007), 107–129.

[11] L. Cherfils and A. Miranville, On the Caginalp system with dynamic boundary conditions and singular potentials, *Appl. Math.* **54** (2009), 89–115.

[12] L. Cherfils, S. Gatti and A. Miranville, A doubly nonlinear parabolic equation with a singular potential, *Discrete Contin. Dyn. Systems S*, **4** (2011), 51–66.

[13] R. Chill, E, Fasangovà and J. Prüss, Convergence to steady states of solutions of the Cahn-Hilliard equation with dynamic boundary conditions, *Math. Nachr.*, **279** (2006), 1448–1462.

[14] C. I. Christov and P. M. Jordan, Heat conduction paradox involving second-sound propagation in moving medis, *Phys. Review Letters,* **94** (2005), 154301.

[15] B. Doumbé, Etude de modèles de champs de phase de type Caginalp, Université de Poitiers, 2013.

[16] A. S. El-Karamany and M. A. Ezzat, On the two-temperature Green-Naghdi thermoelasticity theories, *J. Thermal Stresses*, **34** (2011), 1207–1226.

[17] C. G. Gal and M. Grasselli, The nonisothermal Allen-Cahn equation with dynamic boundary conditions, *Discrete Contin. Dyn. Systems A*, **22** (2008), 1009–1040.

[18] S. Gatti and A. Miranville, Asymptotic behavior of a phase-field system with dynamic boundary conditions, in *Differential Equations: Inverse and Direct Problems (Proceedings of the Workshop "Evolutiob Equations: Inverse and Direct Problems"*, Cortona, June 21-25, 2004), A series of Lecture notes in pure and applied mathematics, **251**, A. Favini and A. Lorenzi eds., Chapman & Hall, 2006, 149–170.

[19] M. Grasselli, A. Miranville, V. Pata and S. Zelik, Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials, *Math. Nachr.*, **280** (2007), 1475–1509.

[20] M. Grasselli, A. Miranville and G. Schimperna, The Caginalp phase-field system with coupled dynamic boundary conditions and singular potentials, *Discrete Contin. Dyn. Systems A*, **28** (2010), 67–98.

[21] M. Grasselli and V. Pata, Existence of a universal attractor for a fully hyperbolic phase-field system, *J. Evol. Eqns.*, **4** (2004), 27–51.

[22] M. Grasselli, H. Petzeltová and G. Schimperna, Long time behavior of solutions to the Caginalp system with singular potential, *Z. Anal. Anwend.*, **25** (2006), 51–72.

[23] A. E. Green and P. M. Naghdi, A re-examination of the basic postulates of thermomechanics, *Proc. Royal Society London A*, **432** (1991), 171–194.

[24] F. Hecht, O. Pironneau, A. Le Hyaric and K. Ohtsuka, Freefem++ Manual, 2012.

[25] J. Jiang, Convergence to equilibrium for a parabolic-hyperbolic phase-field model with Cattaneo heat flux law, *J. Math. Anal. Appl.*, **341** (2008), 149–169.

[26] J. Jiang, Convergence to equilibrium for a fully hyperbolic phase field model with Cattaneo heat flux law, *Math. Methods Appl. Sci.*, **32** (2009), 1156–1182.

[27] A. Miranville, Some mathematical models in phase transition, *Discrete Contin. Dyn. Systems S*, **7** (2014), 271–306.

[28] A. Miranville and R. Quintanilla, A generalization of the Caginalp phase-field system based on the Cattaneo law, *Nonlinear Anal. TMA*, **71** (2009), 2278–2290.

[29] A. Miranville and R. Quintanilla, Some generalizations of the Caginalp phase-field system, *Appl. Anal.*, **88** (2009), 877–894.

[30] A. Miranville and R. Quintanilla, A phase-field model based on a three-phase-lag heat conduction, *Appl. Math. Optim.*, **63** (2011), 133–150.

[31] A. Miranville and R. Quintanilla, A type III phase-field system with a logarithmic potential, *Appl. Math. Letters*, **24** (2011), 1003–1008.

[32] A. Miranville and R. Quintanilla, A generalization of the Allen-Cahn equation, *IMA J. Appl. Math.*, **80** (2015), 410–430.

[33] A. Miranville and R. Quintanilla, A Caginalp phase-field system based on type III heat conduction with two temperatures, *Quart. Appl. Math.*, **74** (2016), 375–398.

[34] A. Miranville and R. Quintanilla, On the Caginalp phase-field systems with two temperatures and the Maxwell-Cattaneo law, *Math. Methods Appl. Sci.*, **39** (2016), 4385–4397.

[35] A. Miranville and S. Zelik, Robust exponential attractors for singularly perturbed phase-field type equations, *Electronic J. Diff. Eqns.,* **2002** (2002), 1–28.

[36] A. Miranville and S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions, *Math. Methods Appl. Sci.*, **28** (2005), 709–735.

[37] R. Quintanilla, A well-posed problem for the three-dual-phase-lag heat conduction, *J. Thermal Stresses*, **32** (2009), 1270–1278.

[38] G. Sadaka, Solution of 2D Boussinesq systems with FreeFem++: The flat bottom case, *J. Numer. Math.*, **20** (2012), 303–324.

[39] H. M. Youssef, Theory of two-temperature-generalized thermoelasticity, *IMA J. Appl. Math.,* **71** (2006), 383–390.

[40] Z. Zhang, Asymptotic behavior of solutions to the phase-field equations with Neumann boundary conditions, *Comm. Pure Appl. Anal.,* **4** (2005), 683–693.

Received January 2018; revised March 2018.

E-mail address: ahmad.makki@math.univ-poitiers.fr
E-mail address: alain.miranville@math.univ-poitiers.fr
E-mail address: georges.sadaka@u-picardie.fr