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NUMERICAL STUDY OF A FAMILY OF DISSIPATIVE KDV EQUATIONS

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Abstract. The weak damped and forced Korteweg-de Vries (KdV) equation on the 1d Torus have been analyzed by Ghidaglia[8, 9], Goubet[10, 11], Rosa and Cabral [3] where asymptotic regularization effects have been proven and observed numerically. In this work, we consider a family of dampings that can be even weaker, particularly it can dissipate very few the high frequencies. We give numerical evidences that point out dissipation of energy, regularization effect and the presence of special solutions that characterize a non trivial dynamics (steady states, time periodic solutions).

1. **Introduction.** In [16], Ott and Sudan have proposed a damped Korteweg-De Vries (KdV) equation as a model of Landau damping for ion acoustic wave, the (linear) damping being nonlocal, and in [17] they have presented different models of damping with the operator $\mathcal{L}$ as

$$u_t + \alpha_1 u_x + \alpha_2 u_{xxx} + \alpha_3 \mathcal{L}(u) = 0, x \in \Omega, t > 0,$$

where $\Omega \subset \mathbb{R}$, $\alpha_i \in \mathbb{R},\ i = 1, 2, 3$ and where $\mathcal{L}$ is the linear damping operator which satisfies

$$\int_\Omega u\mathcal{L}(u)dx \geq 0.$$

The L^2 norm of the solution is then damped as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \alpha_3 \int_\Omega u\mathcal{L}(u)dx = 0.$$

Several authors have studied damped KdV equation, but anyway the literature is not so extensive and particularly relatively very few works have been done on the

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numerical simulation of these models.

The damping models that have been considered are mainly of the form

$$\mathcal{L}(u) = |D|^\alpha u,$$

where $|D| = \sqrt{-\Delta}$, leading to the dissipative KdV equation

$$u_t + u_{xxx} + |D|^\alpha u + uu_x = 0; x \in \Omega, t > 0, \quad (1)$$

$$u(x,0) = u_0(x), \quad (2)$$

with $\alpha \in [0,2]$. Asymptotic estimates of norm decreasing for large time have been obtained by S. Vento [20, 21] when $\Omega = \mathbb{R}$ and in [15] when $\Omega = \mathbb{T}(0,L)$, the torus on $[0,L]$, also simply denoted by $\mathbb{T}$; as a model of dissipative Tsunami wave, D. Dutykh [7] has considered the case $\alpha = 2$ (with the addition of an nonlocal damping in time). Many questions are open, particularly the long time behavior in the periodic case. We here propose to explore numerically some questions related to the time regularization, the asymptotic rate of norm-decreasing and also numerical implementation feature, in the periodic case when using spectral Fourier expansion for the spatial discretization. For that purpose, we consider the family of nonlocal dampings

$$\mathcal{L}_\gamma(u) = \sum_{k\in\mathbb{Z}} \gamma_k \widehat{u}_k e^{\frac{2i\pi kx}{L}}.$$

Here $\widehat{u}_k$ is the k-th Fourier coefficient of u and γ_k are positive real numbers in such a way we have

$$\int_\Omega \mathcal{L}_\gamma(u) u dx = \sum_{k\in\mathbb{Z}} \gamma_k |\widehat{u}_k|^2 \geq 0.$$

This expression of the damping allows to recover number of situations that have been studied, e.g, the choice $\gamma_k = \left|\frac{2\pi k}{L}\right|^\alpha$ corresponds to $\mathcal{L}_\gamma u = |D|^\alpha u$. However, more general cases can be considered and the sequence γ_k can be chosen in such a way to vary the damping frequency by frequency or by band of frequencies.

Let f be a function depending only of the space variable x, we consider the damped and forced KdV equation:

$$u_t + u_{xxx} + \mathcal{L}_\gamma(u) + uu_x = f, \ x \in \mathbb{T}, t > 0, \quad (3)$$

$$u(x,0) = u_0(x). \quad (4)$$

When $\lim_{k\to+\infty} \gamma_k = +\infty$, (e.g. $\gamma_k = k^2$ for a parabolic damping [7]), the equation is regularizing at finite time. When γ_k is constant, say $\mathcal{L}(u) = \gamma u$, the damping is said to be "weak" and is not regularizing at finite time but, as proved by Ghidaglia [8, 9] and Goubet [10, 11], it allows the equation to have a finite dimensional attractor which is in a more regular space than the initial data: this is the asymptotic regularization property. In [3], Rosa and Cabral pointed out a non trivial long time dynamics for mild values of γ, they computed numerically time periodic solutions of various cycle length.

The aim of the present work is to investigate numerically the long time behavior of the forced damped KdV equation (3)-(4) for different type of sequences γ_k with a special focus when $\lim_{k\to+\infty} \gamma_k = 0$: this gives rise to a damping which is more weak than those mentioned before. Indeed, the term $\int_0^L \mathcal{L}_\gamma(u)udx$ which damps the L^2-norm of the solution, cannot be controlled by the L^2 norm of the solution. Since the complete analysis seems very tricky, we first address to these questions numerically. We focus in particular on the following points

- computation of the damping rates for different sequences γ_k
- numerical measure of the Sobolev regularity
- computation of steady state and periodic solution in time

The paper is organized as follows: in Section 2, we give some properties of the equation for general damping operators $\mathcal{L}_\gamma$. Then, in Section 3, we present the discretization schemes and establish some of their properties. Finally, in Section 4 we present number of numerical results with a special focus on the rate of damping, the Sobolev regularization and the computation of special solutions (steady states, time periodic solutions).

2. A model of Damped KdV Equations.

2.1. Problem setting.

When $u \in L^2(\mathbb{T})$, we can consider its Fourier expansion and write

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t) e^{\frac{2i\pi kx}{L}}.$$

We define the nonlocal damping (or dissipative) term as

$$\mathcal{L}_\gamma u = \sum_{k\in\mathbb{Z}} \gamma_k \widehat{u}_k(t) e^{\frac{2i\pi kx}{L}}.$$

For obvious symmetry arguments, we will assume in the sequel of the paper that $\gamma_k \in \mathbb{R}$ and that the relations $\gamma_k = \gamma_{-k}$ are satisfied for each k in $\mathbb{Z}$.
The damped KdV equation we consider here is then

$$u_t + u_{xxx} + \mathcal{L}_\gamma u + uu_x = f, x \in \mathbb{T}, t > 0, \quad (5)$$

$$u(x,0) = u_0(x). \quad (6)$$

As stated in the introduction, we do not address here to the study of the Cauchy problem. The presence of the damping enforces the regularity properties of the solutions and we will have at least the same properties of the Cauchy problem for the non damped KdV problem. The main problem will be the study of the gain of regularity carried by the damping according to the asymptotic behavior of the sequence γ_k.

Due to the particular form of the damping, we need to use some adapted energy space family and we introduce the following notations

Notations:

- $H_\gamma(\mathbb{T}) = \{u \in L^2(\mathbb{T}) / \sum_{k\in\mathbb{Z}} \gamma_k |\widehat{u}_k|^2 < +\infty\}$.
- $\dot{H}_\gamma(\mathbb{T}) = \{u \in H_\gamma(\mathbb{T}) / \int_0^L udx = 0\} = \{u \in H_\gamma(\mathbb{T}) / \widehat{u}_0 = 0\}$.
- Let $(\gamma)_k$ and $(\delta)_k$ be two sequences of positive real numbers. We will note $\gamma \le \delta$ when $\gamma_k \le \delta_k, \ \forall k$.

The associated norm is $|u|_\gamma = \sqrt{\sum_{k\in\mathbb{Z}} \gamma_k |\widehat{u}_k|^2}$.

Finally, we will denote by $< ., . >$ the standard L^2 product.

Remark 1. Let γ_k be a sequence of strictly positive real numbers. It is easy to prove that $H_\gamma(\mathbb{T})$ is a Hilbert space endowed with the scalar product $(u,v)_\gamma = \sum_{k\in\mathbb{Z}} \gamma_k \widehat{u}_k \overline{\widehat{v}_k}$ and of induced norm $|.|_\gamma = \sqrt{(.,.)_\gamma}$.

Let $\alpha \geq \beta$ then $H_\alpha(\mathbb{T}) \hookrightarrow H_\beta(\mathbb{T})$, with continuous injection.
We have also the inequality

$$< u, v > \leq |u|_\gamma |v|_{\frac{1}{\gamma}},\ \forall u \in H_\gamma(\mathbb{T}), \forall v \in H_{\frac{1}{\gamma}}(\mathbb{T})$$

where we have set for convention $(\frac{1}{\gamma})_k = \begin{cases} \frac{1}{\gamma_k} & \text{if } \gamma_k > 0 \\ 0 & \text{else} \end{cases}$.

Remark 2. When $u \in L^2(\mathbb{T})$ and when the sequence γ_k is in ℓ^2 the term $\mathcal{L}_\gamma(u)$ can be identified to the Fourier serie of the convolution product

$$\mathcal{L}_\gamma(u)(x) = \Lambda * u,$$

with $\Lambda(x) = \sum_{k\in\mathbb{Z}} \gamma_k e^{\frac{2i\pi x}{L}}$.

2.2. **Damping properties.** When the KdV equation is nor forced nor damped, it possesses an infinite number of invariants [14], the first ones being

- The mass : $I_0(u) = \int_0^L u(x,t)dx = \int_0^L u_0(x)dx.$
- The L^2 norm : $I_1(u) = \int_0^L (u(x,t))^2 dx = \int_0^L (u_0(x))^2 dx.$
- The Energy : $I_2(u) = \int_0^L \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx - \frac{1}{6}\int_0^L (u(x,t))^3 dx.$

It is then natural to study the effect of the damping on these quantities (the two first in practice). We begin with the mean $I_0(u)$.

2.2.1. *The linear homogeneous equation.* To derive estimates of the propagator associated to the damping, we first consider the linear equation

$$u_t + \mathcal{L}_\gamma(u) + u_{xxx} = 0; x \in \mathbb{T}, t > 0, \tag{7}$$

$$u(x,0) = u_0(x). \tag{8}$$

Assume that $u(x,t) \in L^2(\mathbb{T}), \forall t > 0$. Expanding u in Fourier serie, we get

$$\sum_{k\in\mathbb{Z}} \frac{d\widehat{u}_k(t)}{dt} e^{\frac{2i\pi kx}{L}} + \sum_{k\in\mathbb{Z}} (\gamma_k + \left(\frac{2ik\pi}{L}\right)^3)\widehat{u}_k(t) e^{\frac{2i\pi kx}{L}} = 0. \tag{9}$$

By orthogonality of the trigonometric polynomial, we obtain directly

$$\widehat{u}_k(t) = e^{-(\gamma_k + (\frac{2ik\pi}{L})^3)t}\widehat{u}_k(0). \tag{10}$$

Hence

$$|u|^2_{L^2} = \sum_{k\in\mathbb{Z}} |\widehat{u}_k(t)|^2 = \sum_{k\in\mathbb{Z}} e^{-2\gamma_k t}|\widehat{u}_k(0)|^2. \tag{11}$$

We can now derive bounds.

Proposition 1. *Assume that* $\gamma_k > 0, \forall k \in \mathbb{Z}$ *and that* $u_0 \in H_{\frac{1}{\gamma}}(\mathbb{T})$*. Then*

$$|u|_{L^2}^2 \leq \text{Min}\left(\frac{e^{-1}}{2t}|u_0|_{\frac{1}{\gamma}}^2, |u_0|_{L^2}^2\right), \forall t > 0.$$

More generally, if $u_0 \in H_{\beta/\gamma}(\mathbb{T})$ *then*

$$|u|_{\beta}^2 \leq \frac{e^{-1}}{2t}|u_0|_{\frac{\beta}{\gamma}}^2.$$

Proof. On the one hand we have, taking the scalar product of the equation (7) with u in $L^2(\mathbb{T})$

$$\frac{d|u|_{L^2}^2}{dt} + \sum_{k\in\mathbb{Z}} \gamma_k |\widehat{u}_k(t)|^2 = 0.$$

Hence

$$\frac{d|u|_{L^2}^2}{dt} \leq 0,$$

therefore $|u|_{L^2}^2 \leq |u_0|_{L^2}^2$. On the other hand we have, solving directly the equation (7), mode by mode,

$$|u|_{L^2}^2 = \sum_{k\in\mathbb{Z}} e^{-2\gamma_k t}|\widehat{u}_k(0)|^2 = \sum_{k\in\mathbb{Z}} (\gamma_k e^{-2\gamma_k t})(\frac{1}{\gamma_k}|\widehat{u}_k(0)|^2).$$

The function

$$\gamma \to \gamma e^{-2\gamma t}$$

is uniformly bounded by $\frac{e^{-1}}{2t}$ on $\mathbb{R}^{+*}$. Hence

$$|u|_{L^2}^2 \leq \frac{e^{-1}}{2t} \sum_{k\in\mathbb{Z}} \frac{1}{\gamma_k}|\widehat{u}_k(0)|^2.$$

The second inequality is obtained similarly starting from

$$\beta_k|\widehat{u}_k(t)|^2 = \beta_k e^{-2\gamma_k t}|\widehat{u}_k(0)|^2 = \gamma_k e^{-2\gamma_k t}\left(\frac{\beta_k}{\gamma_k}|\widehat{u}_k(0)|^2\right).$$

Hence the result. □

More generally, using the same elementary proof, we can establish

Proposition 2. *Assume that* $\gamma_k \in [0,1], \forall k \in \mathbb{Z}$ *and that* $u_0 \in H_{\frac{1}{\gamma^s}}$*. Then, for every* $s > 0$

$$|u|_{L^2}^2 \leq \text{Min}\left(e^{-s}\left(\frac{s}{2t}\right)^s |u_0|_{\frac{1}{\gamma^s}}^2, |u_0|_{L^2}^2\right), \forall t > 0.$$

Proof. By bounding uniformly the function $\gamma \to \gamma^s e^{-2\gamma t}$. □

This simple result means that a better decay in time is obtained with higher regularity of the initial data. We can prove that the L^2-norm decreases at a polynomial rate assuming a higher regularity of initial data. We have more precisely

Proposition 3. *Assume that there exist* α*,* β *and* C*, three strictly positive real numbers such that*

i. $|\widehat{u_0}_k|_{L^2}^2 \leq C\gamma_k^{2\delta}$*, with* $\delta = \alpha + \beta$*.*
ii. $\sum_{k\in\mathbb{Z}} \gamma_k^{2\beta} < +\infty$

Then

$$|u|_{L^2}^2 \le Ce^{-1}\left(\frac{\alpha}{t}\right)^{2\alpha}\sum_{k\in\mathbb{Z}}\gamma_k^{2\beta} = \mathcal{O}\left(\frac{1}{t^{2\alpha}}\right)$$

In particular, if $\alpha > 1$, $|u|_{L^2}$ decreases superlinearly in time.

Proof. The proof is similar to the that of the two previous propositions. □

2.2.2. *The nonlinear homogeneous equation.* We take the scalar product in L^2 of the equation

$$u_t + \mathcal{L}_\gamma(u) + u_{xxx} + uu_x = 0;$$

and we obtain

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt} + \sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k(t)|^2 = 0. \tag{12}$$

We have $|u|_\gamma^2 = \sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k(t)|^2$. We can prove the

Proposition 4. *Assume that $\gamma_k > 0, \forall k \in \mathbb{Z}$ and that $u_0 \in L^2(\mathbb{T})$ with $\int_0^L u_0(x)dx = 0$. Assume that $u \in H^3(\mathbb{T}) \cap H_\gamma(\mathbb{T})$ for all $t > 0$. Then $u = 0$ is the only one critical point of the system and is asymptotically stable. More precisely*

i. $\lim_{t\to+\infty}|u|_{L^2} = 0$.

ii. In addition, if $\exists c > 0$ such that $\gamma_k \ge c > 0, \forall k \in \mathbb{Z}$ then $|u|_{L^2} \le e^{-ct}|u|_{L^2}$.

Proof. For the first assertion, we note that

$$\frac{d|u|_{L^2}^2}{dt} \le 0,$$

hence $|u(t)|_{L^2} \le |u_0|_{L^2}, \forall t \ge 0$ and $t \to |u(t)|_{L^2}$ is decreasing in time, consequently $\lim_{t\to\infty}|u(t)|_{L^2}^2 = C$. Assuming $u \in L^2(\mathbb{T}) \cap H_\gamma(\mathbb{T})$, we infer $\lim_{t\to\infty}|u(t)|_\gamma^2 = 0$. Now, since $\gamma_k > 0$ then $\lim_{t\to\infty}\widehat{u}_k = 0$ and $u = 0$ *a.e.* This implies $C = 0$.

Estimates [*ii.*] are obtained by a simple application of a Gronwall lemma: $\gamma_k \ge c > 0$, then $|u|_\gamma \ge c|u|_{L^2}^2$ and we have

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt} + c|u|_{L^2}^2 \le 0,$$

so, by integration, we get directly

$$|u|_{L^2}^2 \le e^{-2ct}|u_0|_{L^2}^2 \text{ therefore } \lim_{t\to+\infty}|u|_{L^2} = 0.$$

□

We have the

Lemma 2.1. *For every sufficiently regular function v (e.g. v is $H^3(\mathbb{T})$ in space), we set $\bar{v}(t) = \frac{1}{L}\int_0^L v(x,t)dx$. Let u be the solution of (5)-(6). Assume that $u \in H^3(\mathbb{T}) \cap H_\gamma(\mathbb{T})$ and that $\bar{f}(t) = 0$, then $\bar{u}(t) = e^{-\gamma_0 t}\bar{u}(0)$*

Proof. We have

$$\int_0^L u_{xxx}(x,t)dx = 0 \text{ and } \int_0^L uu_x(x,t)dx = \frac{1}{2}\int_0^L \frac{\partial}{\partial x}u^2(x,t)dx = 0.$$

Hence integrating each term of the equation on space on the interval $[0, L]$, we get

$$\int_0^L \left(\frac{\partial u}{\partial t} + \mathcal{L}_\gamma(u)\right) dx = 0.$$

But $\displaystyle\int_0^L \mathcal{L}_\gamma(u)(x,t)dx = \int_0^L \sum_{k\in\mathbb{Z}} \gamma_k \widehat{u}_k e^{\frac{2i\pi x}{L}} dx = L\gamma_0\widehat{u}_0 = \gamma_0 \int_0^L udx.$

$$\text{Therefore } \frac{d}{dt}\frac{1}{L}\int_0^L u(x,t)dx + \frac{1}{L}\int_0^L \mathcal{L}_\gamma(u)(x,t)dx = 0,$$

so

$$\frac{d\bar{u}}{dt} + \gamma_0\bar{u} = 0,$$

Hence the result. In particular, if $\bar{u}(0) = 0$ then $\bar{u}(t) = 0$, $\forall t \geq 0$. □

Corollary 1. *Assume that* $u \in H^3(\mathbb{T}) \cap H_\gamma(\mathbb{T})$ *and that* $u_0 \in H^3(\mathbb{T})$*. Assume in addition that* $\gamma_k > 0, \forall k \in \mathbb{Z}^*$*,* $\gamma_0 = 0$ *and* $u_0 \in L^2(\mathbb{T})$*. Then*

$$\lim_{t\to+\infty} u(t) = \widehat{u}_0 = \frac{1}{L}\int_0^L u_0(x)dx.$$

Proof. It suffices to introduce $v = u - \widehat{u}_0$ and to combine Lemma 2.1 and proposition 4. □

When the γ_k are not bounded from below, the orbit converges to 0 in L^2 norm, but it can be at an arbitrary slow rate: it depends on how γ_k converge to 0 as k goes to infinity.

The main difficulty comes out from the fact that, when γ_k converge toward 0 as $|k|$ goes to infinity, there is no injection from $H_\gamma(\mathbb{T})$ to $L^2(\mathbb{T})$. So, the ratio of the two associated norm of the solution plays an important role as pointed out hereafter. We introduce the function

$$G \;:\; (u,t) \mapsto G(u,t) = \frac{|u|_\gamma}{|u|_{L^2}} = \sqrt{\frac{\sum_{k\in\mathbb{Z}} \gamma_k |\widehat{u}_k|^2}{\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2}},$$

and in practice we shall use $G(t)$ for $G(u,t)$ when there will be no ambiguity.

Proposition 5. *Let* $u(x,t)$ *be a regular solution of the homogeneous equation (5). We assume that* $G(u,t)$ *is* $\mathcal{C}^1$ *in* t*. Then we have the estimates*

- $|u(t)|^2_{L^2} = e^{-\int_0^t G^2(s)ds} |u_0|^2_{L^2}$
- $|u(t)|^2_\gamma = G^2(t) e^{-\int_0^t G^2(s)ds} |u_0|^2_{L^2}$

In particular, $\lim_{t\to+\infty} |u|_{L^2} = 0$ *iff* $t \mapsto G(t) \notin L^2_t(0, +\infty)$.

Proof. As above, taking the scalar product of the equation (5) in $L^2(\mathbb{T})$ with u, we get

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt}+|u|_\gamma^2=0.$$

We have by definition of G : $|u|_\gamma^2=G^2(t)|u|_{L^2}^2$. Hence

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt}+G^2(t)|u|_{L^2}^2=0,$$

from that we infer directly

$$|u(t)|_{L^2}^2=e^{-2\displaystyle\int_0^t G^2(s)ds}|u_0|_{L^2}^2. \tag{13}$$

So $\lim\limits_{t\to\infty}|u|_{L^2}=0$ iff $t\mapsto G(t)\notin L_t^2(0,+\infty)$. Estimate in H_γ norm follows directly, we have

$$|u(t)|_\gamma^2=G^2(t)e^{-2\displaystyle\int_0^t G^2(s)ds}|u_0|_{L^2}^2. \tag{14}$$

We deduce from the proposition 4 that, since $\lim\limits_{t\to\infty}|u|_\gamma=0$, then

$$\lim_{t\to\infty}G^2(t)e^{-\displaystyle\int_0^t G^2(s)ds}=0.$$

∎

The condition $G(t)\notin L_t^2(0,+\infty)$ is of course automatically satisfied when $H_\gamma(\mathbb{T})\subset L^2(\mathbb{T})$ with continuous injection or, equivalently when γ_k are uniformly bounded by below by a strictly nonnegative constant. In such a case, $G(t)$ is bounded from below and $|u|_{L^2}$ converges to zero at an exponential rate as t goes to infinity. In the opposite case, say when $\lim\limits_{k\to\infty}\gamma_k=0$, according to the previous proposition, we still have $G(t)\notin L_t^2(0,+\infty)$ but we can not obtain a rate of convergence of $|u|_{L^2}$ to 0. These points are crucial for proving the existence of bounded absorbing sets, see [19].

2.2.3. *The nonlinear forced equation.* Our goal here is to explore numerically the long time behavior of the solutions of the forced and damped equation, and particularly to look to the possible nontrivial dynamics in that case.

First, and as before, we derive estimates in L^2 and H_γ norms:

Proposition 6. *Assume that f belongs to $H_{\frac{1}{\gamma}}(\mathbb{T})$ and that $u_0\in L^2(\mathbb{T})\bigcap H_\gamma(\mathbb{T})$. Then*

$$|u(t)|_{L^2}^2\le e^{-\displaystyle\int_0^t G^2(s)ds}|u_0|_{L^2}^2+\int_0^t e^{-\displaystyle\int_s^t G^2(\tau)d\tau}|f|_{\frac{1}{\gamma}}^2ds.$$

Proof. Taking the scalar product of the equation (5) in L^2 with u, we get

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt}+|u|_\gamma^2=<f,u>.$$

Now,

$$|<f,u>|\le|f|_{\frac{1}{\gamma}}|u|_\gamma.$$

Hence, introducing $G(t)$ and using Young's inequality,

$$\frac{1}{2}\frac{d|u|_{L^2}^2}{dt} + G^2(t)|u|_{L^2}^2 \leq \frac{1}{2\varepsilon}|f|_{\frac{1}{\gamma}}^2 + \frac{\varepsilon}{2}|u|_{\gamma}^2 = \frac{1}{2\varepsilon}|f|_{\frac{1}{\gamma}}^2 + \frac{\varepsilon}{2}G^2(t)|u|_{L^2}^2,$$

for any $\varepsilon > 0$. With $\varepsilon = 1$ we obtain, by Gronwall's lemma

$$|u(t)|_{L^2}^2 \leq e^{-\int_0^t G^2(s)ds}|u_0|_{L^2}^2 + \int_0^t e^{-\int_s^t G^2(\tau)d\tau}|f|_{\frac{1}{\gamma}}^2 ds. \tag{15}$$

In a same way, we derive estimates in $H_\gamma(\mathbb{T})$, by multiplying each term by $G^2(t)$

$$|u(t)|_{\gamma}^2 \leq G^2(t)e^{-\int_0^t G^2(s)ds}|u_0|_{L^2}^2 + \int_0^t e^{-\int_s^t G^2(\tau)d\tau}G^2(t)|f|_{\frac{1}{\gamma}}^2 ds. \tag{16}$$

□

We deduce immediately the following result:

Corollary 2. *We make the assumptions of the previous proposition. In addition, we assume that the function*

$$F : t \mapsto \int_0^t e^{-\int_s^t G^2(\tau)d\tau} ds$$

is uniformly bounded in t. *Then the equation possesses a bounded absorbing set in* L^2.

3. Numerical schemes.

3.1. Discretization in space.

We will use a classical Fourier pseudo-spectral spatial discretization such as described in [12]. To complete the discretization we need to define well suited time marching schemes, as follows.

3.2. Time marching schemes.

We present hereafter 4 implicit/semi-implicit time marching schemes for the forced and damped KdV equation:

- Forward Euler's
- Crank Nicolson
- Sanz-Serna
- Splitting
- Strang Splitting

We write them considering only the semi discretization in time and give for 3 of them L^2 stability results which remain valid when finite dimension approximation in space is considered: this is simply due to orthogonality property of the interpolation trigonometric polynomials.

Finally, we need to solve numerically at each time step a fixed point problem. It can be done by using the classical Picard iterate. In some situations, e.g. for Sanz-Serna, the Picard fixed point method needs a small time step Δt to converge: this is artificial since the scheme is unconditionally L^2-stable. Such drawback can be overcome by using other fixed point solver which has a better stability, as proposed in [1], see also subsection 3.2.5 below.

3.2.1. *Forward Euler.* As first numerical scheme, we propose

$$\frac{u^{(n+1)}-u^{(n)}}{\Delta t}+\mathcal{L}_\gamma u^{(n+1)}+\mathcal{D}^3u^{(n+1)}+\frac{1}{2}\mathcal{D}(u^{(n+1)})^2=f. \tag{17}$$

Here $\mathcal{D}$ is the skew symmetric linear operator of the first spatial derivative or of its discretization. We prove the following result:

Proposition 7. *Assume that $u^{(0)}=u_0\in L^2(\mathbb{T})$ and $f\in H_{\frac{1}{\gamma}}(\mathbb{T})$. Then the sequence $u^{(n)}$ generated by the Forward Euler scheme is well defined, belongs in $L^2(\mathbb{T})$ and*

$$|u^{(n+1)}|_2^2+|u^{(n+1)}-u^{(n)}|_2^2+\Delta t\sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k^{(n+1)}|^2\le|u^{(n)}|_2^2+\Delta t\sum_{k\in\mathbb{Z}}\frac{1}{\gamma_k}|\widehat{f}_k|^2.$$

In addition if $f=0$, then $\lim\limits_{n\to+\infty}|u^{(n)}|_{L^2}=0$.

Proof. Using the identity

$$<u^{(n+1)}-u^{(n)},u^{(n+1)}>=-\frac{1}{2}\left(|u^{(n)}|_2^2-|u^{(n+1)}-u^{(n)}|_2^2-|u^{(n+1)}|_2^2\right),$$

we obtain after usual computations

$$|u^{(n+1)}|_2^2+|u^{(n+1)}-u^{(n)}|_2^2+2\Delta t<\mathcal{L}_\gamma u^{(n+1)},u^{(n+1)}>=|u^{(n)}|_2^2+2\Delta t<u^{(n+1)},f>, \tag{18}$$

where

$$<\mathcal{L}_\gamma(u)^{(n+1)},u^{(n+1)}>=\sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k^{(n+1)}|^2\ge 0.$$

At this point, we use the Young's inequality

$$<u^{(n+1)},f>\le\frac{\varepsilon}{2}\sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k^{(n+1)}|^2+\frac{1}{2\varepsilon}\sum_{k\in\mathbb{Z}}\frac{1}{\gamma_k}|\widehat{f}_k|^2,$$

for $\varepsilon>0$ that will be fixed later on. Hence

$$|u^{(n+1)}|_2^2+|u^{(n+1)}-u^{(n)}|_2^2+2\Delta t(1-\frac{\varepsilon}{2})\sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k^{(n+1)}|^2\le|u^{(n)}|_2^2+2\Delta t\frac{1}{2\varepsilon}\sum_{k\in\mathbb{Z}}\frac{1}{\gamma_k}|\widehat{f}_k|^2.$$

Hence the result with $\epsilon=1$.

If $f=0$, the identity (18) becomes

$$|u^{(n+1)}|_2^2+|u^{(n+1)}-u^{(n)}|_2^2+2\Delta t|u^{(n+1)}|_\gamma^2=|u^{(n)}|_2^2 \tag{19}$$

Hence, the sequence $|u^{(n)}|_2$ is decreasing and bounded from below (by 0), then convergent to C. It follows that

$$\lim_{n\to+\infty}\sum_{k\in\mathbb{Z}}\gamma_k|\widehat{u}_k^{(n)}|^2=0.$$

Therefore, since $\gamma_k>0$, we have $\lim\limits_{n\to+\infty}\widehat{u}_k^n=0$ and then $C=0$. □

We give also a discrete version of proposition 5:

Proposition 8. *Let $u^{(n)}$ the sequence generated by the Forward Euler scheme. We assume that $f=0$. We set $G^{(n)}=\dfrac{|u^{(n)}|_\gamma}{|u^{(n)}|_{L^2}}$. We have*

$$|u^{(n)}|_{L^2}^2\le\left(\prod_{j=1}^{n}\frac{1}{1+2\Delta t(G^{(j)})^2}\right)|u_0|_{L^2}^2.$$

In addition, for Δt small enough, if $(G^{(j)})_{j\in\mathbb{Z}} \notin \ell^2$, then $\lim_{n\to+\infty} |u^{(n)}|_{L^2} = 0$.

Proof. Taking the scalar product in L^2 with $u^{(n+1)}$ we obtain

$$\frac{1}{2\Delta t}\left(|u^{(n+1)}|_{L^2}^2 - |u^{(n)}|_{L^2}^2 + |u^{(n+1)} - u^{(n)}|_{L^2}^2\right) + (G^{(n+1)})^2|u^{(n+1)}|_{L^2}^2 = 0.$$

Therefore

$$(1 + 2\Delta t(G^{(n+1)})^2)|u^{(n+1)}|_{L^2}^2 + |u^{(n+1)} - u^{(n)}|_{L^2}^2 \le |u^{(n)}|_{L^2}^2.$$

Now, for Δt small enough we have

$$\log\left(\prod_{j=1}^{n}\frac{1}{1+2\Delta t(G^{(j)})^2}\right) = -\sum_{j=1}^{n}\log(1+2\Delta t(G^{(j)})^2) \simeq -2\Delta t\sum_{j=1}^{n}(G^{(j)})^2.$$

Hence, if $G^{(j)} \notin \ell^2$ then $\lim_{n\to+\infty} |u^{(n)}|_{L^2} = 0$. □

3.2.2. *Crank-Nicolson Scheme.* The classical Crank-Nicolson scheme writes as

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} + \mathcal{L}_\gamma\frac{u^{(n+1)} + u^{(n)}}{2} + \mathcal{D}^3\frac{u^{(n+1)} + u^{(n)}}{2} + \frac{1}{4}\mathcal{D}\left((u^{(n+1)})^2 + (u^{(n)})^2\right) = f, \quad (20)$$

with the notations as above. This scheme is second order accurate in time. Here we do not have

$$\left\langle \mathcal{D}\left((u^{(n+1)})^2 + (u^{(n)})^2\right), \frac{u^{(n+1)} + u^{(n)}}{2}\right\rangle = 0,$$

so we can not derive uniform L^2 bounds for $u^{(n)}$ from the scheme. However is practice, this scheme gives satisfactory numerical results, see section 4.

3.2.3. *Sanz-Serna Scheme.* The Sanz-Serna scheme is second order accurate in time and corresponds to a mid point quadrature formula in the evaluation of the vector field, see [18]. It writes here as

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} + \mathcal{L}_\gamma\frac{u^{(n+1)} + u^{(n)}}{2} + \mathcal{D}^3\frac{u^{(n+1)} + u^{(n)}}{2} + \frac{1}{2}\mathcal{D}\left(\frac{u^{(n+1)} + u^{(n)}}{2}\right)^2 = f, \quad (21)$$

with always the same the notations.

Proposition 9. *Assume that $u_0 \in L^2(\mathbb{T})$ and $f \in L^2(\mathbb{T})\bigcap H_\gamma(\mathbb{T})$. Then the scheme (21) is stable in $L^2(\mathbb{T})$ for all $\Delta t > 0$.*

Proof. We take the L^2 scalar product of each term of (21) with $\dfrac{u^{(n+1)} + u^{(n)}}{2}$ and obtain

$$\frac{|u^{(n+1)}|_{L^2}^2 - |u^{(n)}|_{L^2}^2}{2\Delta t} + \frac{1}{4}|u^{(n+1)} + u^{(n)}|_\gamma^2 = \left\langle f, \frac{u^{(n+1)} + u^{(n)}}{2}\right\rangle$$

So, using duality and Young's inequality, we have

$$\frac{|u^{(n+1)}|_{L^2}^2 - |u^{(n)}|_{L^2}^2}{2\Delta t} + \frac{1}{4}|u^{(n+1)} + u^{(n)}|_\gamma^2 \le \frac{1}{2}|f|_{\frac{1}{\gamma}}^2 + \frac{1}{2}\left|\frac{u^{(n+1)} + u^{(n)}}{2}\right|_\gamma^2.$$

Finally, after the usual simplifications

$$|u^{(n+1)}|_{L^2}^2 + \frac{\Delta t}{4}|u^{(n+1)} + u^{(n)}|_\gamma^2 \le |u^{(n)}|_{L^2}^2 + \Delta t|f|_{\frac{1}{\gamma}}^2.$$

Hence the L^2 stability on every time interval $[0, T]$. □

The scheme (21) is indeed second order accurate and L^2 stable but we can not establish contraction properties as for Forward Euler's which is only first order accurate in time. Indeed, with $f = 0$ we only obtain the relation

$$|u^{(n+1)}|_{L^2}^2 + \frac{\Delta t}{4}|u^{(n+1)} + u^{(n)}|_\gamma^2 = |u^{(n)}|_{L^2}^2$$

which implies that $|u^{(n)}|_{L^2}$ is decreasing (then convergent) but we cannot conclude that the limit is 0. In order to establish both accuracy and contraction properties we consider in the sequel a scheme based on the Strang Splitting.

3.2.4. *A Strang splitting time scheme.* The main idea of a time splitting scheme is to treat separately the time integration in the one hand of pure dispersive part of the equation and the damped, and on the other hand of the damped part. Of course this last one is less expensive in computations, so a natural approach is to apply the classical Strang splitting as follows (denoted by $SpSt$ in the sequel):

$$u^{(n+1/3)} = e^{-\frac{\Delta t}{2}\mathcal{L}_\gamma}u^{(n)}, \quad (22)$$

$$\frac{u^{(n+2/3)} - u^{(n+1/3)}}{\Delta t} + \frac{1}{2}\mathcal{D}^3\left(u^{(n+2/3)} + u^{(n+1/3)}\right) + \frac{1}{8}\mathcal{D}((u^{(n+2/3)} + u^{(n+1/3)})^2) = f, \quad (23)$$

$$u^{(n+1)} = e^{-\frac{\Delta t}{2}\mathcal{L}_\gamma}u^{(n+2/3)}. \quad (24)$$

Here, the operator $\mathcal{S}_\gamma = e^{-\frac{\Delta t}{2}\mathcal{L}_\gamma}$ is for every $u \in L^2(\mathbb{T})$ as

$$\mathcal{S}_\gamma u = \sum_{k\in\mathbb{Z}} e^{-\frac{\Delta t}{2}\mathcal{L}_{\gamma_k}}\widehat{u}_k e^{\frac{2i\pi kx}{L}}.$$

When using an exact time integration, the Strang splitting is second order time accurate. Steps (22) and (24) correspond to exact integration while (23) is a Sanz Serna's which is second order accurate. The resulting so, as we will check numerically. At this point we derive stability bounds

Proposition 10. *Assume that $\gamma_k > 0, \forall k$ and $u^{(0)} \in L^2(\mathbb{T})$. Then the sequence $u^{(n)}$ generated by the scheme (22)-(23)-(24) is well defined and the scheme is unconditionnally stable in L^2. If $f = 0$ then $\lim_{n\to+\infty}|u^{(n)}|_{L^2} = 0$. We have in addition the following estimates*

- *if $\exists c > 0,\ \gamma_k \geq c > 0, \forall k$, then*

$$|u^{(n)}|_{L^2} \leq \delta^{2n}|u^{(0)}|_{L^2} + \Delta t\delta\frac{1}{1-\delta^2}|f|_{L^2}$$

  with $\delta = e^{-\frac{\Delta tc}{2}}$
- *if $u^{(0)} \in L^2(\mathbb{T}) \cap H_{\frac{1}{\gamma}}$ and $f = 0$ then*

$$|u^{(n)}| \leq \frac{e^{-1}}{N\Delta t}|u^{(0)}|_{\frac{1}{\gamma}}.$$

Proof. We first assume that $\gamma_k \geq \gamma = cst, \forall k \in \mathbb{Z}$. We have then $|\mathcal{S}_\gamma u|_{L^2} \leq e^{-\frac{\Delta t\gamma}{2}}|u|_{L^2} = \delta|u|_{L^2}$. We obtain directly the relations

$$|u^{(n+1/3)}|_{L^2} \leq \delta|u^{(n)}|_{L^2} \text{ and } |u^{(n+1)}|_{L^2} \leq \delta|u^{(n+2/3)}|_{L^2}.$$

Taking the scalar product in $L^2(\mathbb{T})$ of each term of (23) with $u^{(n+2/3)} + u^{(n+1/3)}$, we get

$$|u^{(n+2/3)}|_{L^2}^2 - |u^{(n+1/3)}|_{L^2}^2 \leq \Delta t < f, u^{(n+2/3)} + u^{(n+1/3)} > \leq \Delta t|f|\left(|u^{(n+2/3)}|_{L^2} + |u^{(n+1/3)}|_{L^2}\right),$$

therefore

$$|u^{(n+2/3)}|_{L^2} - |u^{(n+1/3)}|_{L^2} \leq \Delta t |f|_{L^2}.$$

Finally

$$|u^{(n+1)}|_{L^2} \leq \delta |u^{(n+2/3)}|_{L^2} \leq \delta \left(\delta |u^{(n)}|_{L^2} + \Delta t |f|_{L^2} \right).$$

In summary

$$|u^{(n+1)}|_{L^2} \leq \delta^2 |u^{(n)}|_{L^2} + \Delta t \delta |f|_{L^2},$$

and by induction we find

$$|u^{(n)}|_{L^2} \leq \delta^{2n} |u^{(0)}|_{L^2} + \Delta t \delta \frac{1}{1-\delta^2} |f|_{L^2},$$

hence the uniform L^2 stability. The sequence $u^{(n)}$ is well defined in $L^2(\mathbb{T})$. Also, we infer from the previous relation that if $f = 0$ then $\lim_{n \to +\infty} |u^{(n)}|_{L^2} = 0$.

Let us now study the general case $\gamma_k > 0$, with possibly $\lim_{k \to +\infty} \gamma_k = 0$. We first need to show that

$$\lim_{k \to +\infty} |\mathcal{S}_\gamma^N v|_{L^2} = 0$$

for any $v \in L^2(\mathbb{T})$. We have the following result

Lemma 3.1. *Let $v \in L^2(\mathbb{T})$. Then*

$$\lim_{N \to +\infty} |\mathcal{S}_\gamma^N v|_{L^2} = 0.$$

In addition, if $v \in L^2(\mathbb{T}) \cap H_{\frac{1}{\gamma}}(\mathbb{T})$ we have the estimate

$$|\mathcal{S}_\gamma^N v|_{L^2} \leq 2 \frac{e^{-1}}{N \Delta t} |v|_{\frac{1}{\gamma}}.$$

Proof. Since $v \in L^2(\mathbb{T})$, for a given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\forall N \geq N_1, \sum_{|k| > N_1} |\widehat{v}_k|^2 < \frac{\varepsilon}{\sqrt{2} M}.$$

where we have set $M = |v|_{L^2}$. Now, we write $\mathcal{S}^N v$ as

$$|\mathcal{S}^N v|_{L^2}^2 = \sum_{|k| \leq N_1} e^{-N \Delta t \gamma_k} |\widehat{v}_k|^2 + \sum_{|k| > N_1} e^{-N \Delta t \gamma_k} |\widehat{v}_k|^2$$

The second part is bounded by $\frac{\varepsilon^2}{2}$. Now, since N_1 is fixed, we define $\underline{\gamma}$ as $\underline{\gamma} = \inf_{|k| \leq N_1} \gamma_k > 0$. We can write

$$\sum_{|k| \leq N_1} e^{-N \Delta t \gamma_k} |\widehat{v}_k|^2 \leq e^{-N \Delta t \underline{\gamma}} M^2.$$

Finally for the same $\varepsilon > 0$ there exists N_2 such that for every $N \geq N_2$, $e^{-N \Delta t \underline{\gamma}} < \frac{\varepsilon^2}{2M^2}$. Summing these inequalities, we obtain the result. Notice that we do not have any estimate of the rate of convergence; the rate depends on both v and γ.

Assume now that $v \in L^2(\mathbb{T}) \cap H_{\frac{1}{\gamma}}$. Proceeding exactly as in Proposition 1, we find

$$|\mathcal{S}_\gamma^N v|_{L^2} \leq \frac{e^{-1}}{N \Delta t} |v|_{\frac{1}{\gamma}}.$$

This achieves the proof of the lemma. □

Let us now turn back to the time marching scheme. We have the relations

$$u^{(n+1/3)} = \mathcal{S}_\gamma u^{(n)}, \tag{25}$$

$$|u^{(n+2/3)}|_{L^2} \leq |u^{(n+1/3)}|_{L^2} + \Delta t |f|_{L^2}, \tag{26}$$

$$u^{(n+1)} = \mathcal{S}_\gamma u^{(n+2/3)}. \tag{27}$$

Hence,

$$|u^{(n+1)}|_{L^2} \leq \Delta t |f|_{L^2} + |\mathcal{S} u^{(n)}|_{L^2}.$$

The L^2-stability follows by induction since $|\mathcal{S}_\gamma u^{(n)}|_{L^2} \leq |u^{(n)}|_{L^2}$. Now, if $f = 0$, we have directly

$$|u^{(n)}|_{L^2} \leq |\mathcal{S}^n u^{(0)}|_{L^2}$$

and we conclude with the previous lemma. □

This splitting scheme combines thus both second order accuracy in time (as Crank-Nicolson's or Sanz-Serna's) and similar L^2 stability properties as Forward Euler's. We illustrate these properties in the numerical simulations.

Remark 3. The splitting scheme (that we note Sp)

$$\frac{u^{(n+1/3)} - u^{(n)}}{\Delta t/2} + \mathcal{L}_\gamma u^{(n+1/3)} = 0, \tag{28}$$

$$\frac{u^{(n+2/3)} - u^{(n+1/3)}}{\Delta t} + \mathcal{D}^3 u^{(n+2/3)} + \frac{1}{2}\mathcal{D}(u^{(n+2/3)})^2 = f, \tag{29}$$

$$\frac{u^{(n+1)} - u^{(n+2/3)}}{\Delta t/2} + \mathcal{L}_\gamma u^{(n+1)} = 0, \tag{30}$$

allows to recover also L^2 stability properties. Assuming that $f \in H_{\frac{1}{\gamma}} \cap L^2$, we can show that

$$|u^{(n)}|^2_{L^2} + \Delta t \sum_{j=1}^{n} |u^{(j)}|^2_\gamma \leq |u_0|^2_{L^2} + 2T \left(|f|^2_{L^2} + |f|^2_{\frac{1}{\gamma}} \right), \ \forall n \in \mathbb{N}.$$

However, it is only first order accurate all the three steps being only first order accurate in time, see Section 4 hereafter.

3.2.5. *Implementation.* The schemes presented above are implicit or semi implicit and a fixed point problem must be solved at each iteration that can be written as

$$u^{(n+1)} = \Phi(\Delta t, u^{(n)}, u^{(n+1)}), \tag{31}$$

where the definition of Φ depends on the chosen time marching scheme. The simplest method to solve (31) is Picard's iterate:

```
For n=0 ···
Set        v^(0) = u^(n), m = 0
           while ||φ(Δt, u^(n), v^(m)) − v^(m)|| > ε
           v^(m+1) = Φ(Δt, u^(n), v^(m))
           m = m + 1
           End while
Set        u^(n+1) = v^(m)
End for
```

Here $\varepsilon > 0$ is a small parameter fixed, e.g. $\varepsilon = 1.e - 12$. In all the simulation we made the classical Picard iterate was sufficient to converge in very few iterations.

4. **Numerical results.** The results presented here have been obtained using Matlab software. We have used the Crank-Nicolson scheme together with the standard Picard iterate to solve the fixed point at each step. The other time schemes (Euler's, Sanz-Serna, Strang Splitting Sp and $SpSt$) give comparable results.

We now apply this technique to the forced damped KdV equation for different sequences γ_k and more precisely

- Constant damping $\gamma_k = 1, \forall k$.
- Band limited damping: $\gamma_k = \chi_{k_1 \le k \le k_2} = \begin{cases} 1 & \text{if } k_1 \le k \le k_2 \\ 0 & \text{otherwise} \end{cases}$.
- Comb-like damping: $\gamma_k = \begin{cases} 1 \text{ if } k \text{ is even} \\ 0 \text{ otherwise} \end{cases}$.
- $\gamma_k > 0$ with $\lim\limits_{k \to +\infty} \gamma_k = 0$, such as $\gamma_k = \dfrac{1}{(1+|k|)^\alpha}$, with $\alpha = 1/4, 1, 2, \cdots$

At this point we justify the choice of the damping coefficients γ_k. When γ_k decreases very slowly to 0, e.g. for $\gamma_k = 1/\ln(1+|k|)$, the L^2 norm of the solution converges toward 0 as t goes to $+\infty$ at a rate comparable to that observed with $\gamma_k = 1$; in the opposite case, when γ_k decreases fastly to 0, e.g. for $\gamma_k = e^{-|k|}$, the damping is very weak and the decreasing of the L^2-norm is very slow, one has to integrate on a long time interval to observe it. However, in the both cases, the agreement with the theory is complete: when $\gamma_k > 0$ $\lim\limits_{t \to +\infty} |u|_{L^2} = 0$. The rate of the decay depends on how γ_k converge to 0 as $k \to +\infty$. For these reasons, we chose to display mild-situations, say polynomial decay of γ_k ($\gamma_k = \dfrac{1}{(1+|k|)^\alpha}$) because it allows to tune the rate of the damping.
We obtained the numerical results by considering the following different initial data

$u_0(x) = S1 = 3c\text{sech}(\sqrt{\frac{c}{2}}(x-pL))^2$, with $c = 1, p = 0.4$ is the soliton
$u_0(x) = S2 = \chi_{0.4L<x<0.6L}$ corresponds to the crennel
$u_0(x) = S3 = \sin(\frac{2\pi x}{L})$is the sine data
$u_0(x) = S4 = 50\chi_{x>\pi}\sin(4x)$(this is the initial data to compute time-periodic solutions for $L = 2\pi$ (inspired by that use by [3]))

Unless specified, for all the numerical simulations, we work with $L = 100$, $N = 2^9$ and $\Delta t = 0.0005$.

We first illustrate the effect of the damping for different sequences γ_k on both the linear and the nonlinear equation. At first, we illustrate the damping effect in the two energy norms $|.|_{L^2}$ and $|.|_\gamma$. We see in particular that, as expected, when $\gamma_k > 0$ the solution of the homogeneous equation converges toward 0 in $L^2(\mathbb{T}) \bigcap H_\gamma(\mathbb{T})$. However, when $\lim\limits_{k \to +\infty} \gamma_k = 0$ the rate of convergence depends on u_0 and on its Fourier decomposition.

4.1. **Accuracy of the schemes.** In Figure 1 below we compare the accuracy of the various schemes presented in the previous section; as expected Euler and Sp are first order while the others (C-N, Sanz-Serna and $SpSt$) are second order accurate.

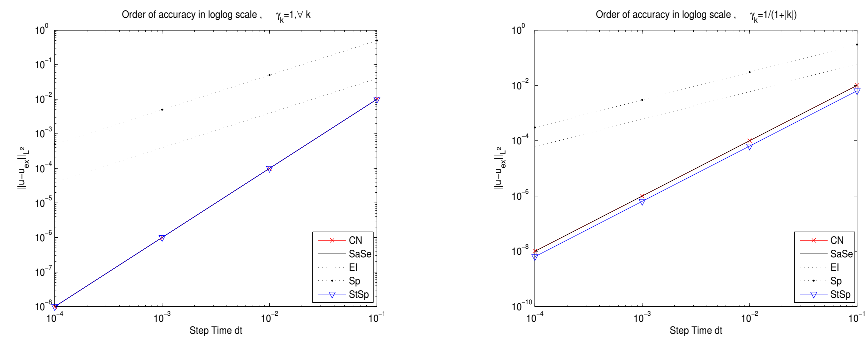


FIGURE 1. Comparison of the rate of the accuracy of the schemes for $\gamma_k = 1, \forall k$ (left) and for $\gamma_k = \dfrac{1}{1+|k|}$ (right).

At this point we would like to illustrate the necessity to assume $\gamma_k > 0$ to obtain damping of the L^2 and the H_γ norms in time. We recall the identity

$$\widehat{\partial_x u^2}_k = \widehat{v}_k = \frac{2i\pi k}{L} \sum_{n\in\mathbb{Z}} \widehat{u}_{k-n}\widehat{u}_n.$$

Hence, if $\widehat{u}_{2p+1} = 0$, for $p \in \mathbb{Z}$ then $\widehat{v}_{2k+1} = 0$. Therefore, it is easy to show that if $\widehat{u}_0$ and f have only even nonzero frequency, then the sequence generated by the different schemes enjoy of the same property. Consequently a comb damping supported by even frequency will damp all the solution while a comb-damping supported only by odd frequency will have no damping effect. Indeed let $u \in L^2$ such that $\widehat{u}_{2k+1} = 0, \forall k \in \mathbb{Z}$. We set $v = \partial_x u^2$. We have

$$\widehat{v}_k = \frac{2i\pi k}{L} \sum_{n\in\mathbb{Z}} \widehat{u}_{k-n}\widehat{u}_n = \frac{2i\pi k}{L}\left(\sum_{n\in\mathbb{Z}} \widehat{u}_{k-2n}\widehat{u}_{2n} + \sum_{n\in\mathbb{Z}} \widehat{u}_{k-2n-1}\widehat{u}_{2n+1} \right).$$

If $k = 2p+1$, we have

$$\widehat{v}_{2p+1} = \frac{2i\pi k}{L}\left(\sum_{n\in\mathbb{Z}} \widehat{u}_{2p+1-2n}\widehat{u}_{2n} + \sum_{n\in\mathbb{Z}} \widehat{u}_{2p+1-2n-1}\widehat{u}_{2n+1} \right) = 0.$$

More generally, we can prove the following result

Lemma 4.1. *Let $u, v \in L^\infty(\mathbb{T})$. Assume that $\widehat{u}_{2k+1} = \widehat{v}_{2k+1} = 0$. Then*

$$\widehat{uv}_{2k+1} = 0, \ \ k \in \mathbb{Z}.$$

It follows by induction that if $\widehat{u}_{2k+1} = 0$, then $\widehat{u}^p_{2k+1} = 0$.
As an illustration, in Figures 2 and 3, we show building counterexample that it is indeed necessary to take $\gamma_k > 0, \forall k$ to have a damping in L^2 norm.

4.2. **Homogeneous KdV.** In Figures 4 to 9 we observe a perfect agreement with the results established above in the homogeneous case: when $\gamma_k > 0, \forall k$ the solution converges to 0 in $L^2(\mathbb{T})$ and $H_\gamma(\mathbb{T})$ norms, however, when $\lim_{k\to+\infty} \gamma_k = 0$ the rate (slope in log scale) depends both on the initial data u_0 and on γ_k; the case of a bandpass, i.e., when $\gamma_k = 0$ for $N_1 \le |k| \le N_2$, is reported in [6] and we observe according to the cases convergence to 0 if enough frequencies are damped but this is not the case when the damping is not sufficient.

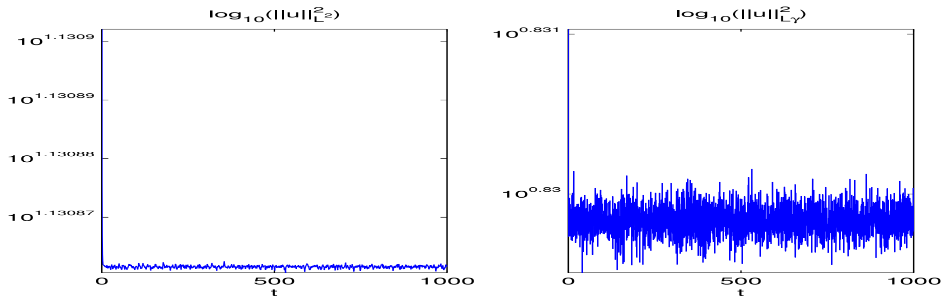


FIGURE 2. $f = 0$, $\widehat{u_0}_{2k+1} = 0$ and $\gamma_{2k} = 0, \gamma_{2k+1} = 1$

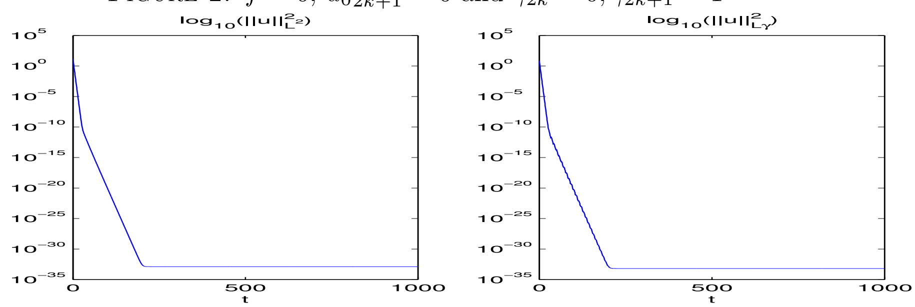


FIGURE 3. $f = 0$, $\widehat{u_0}_{2k+1} = 0$ and $\gamma_{2k+1} = 0, \gamma_{2k} = 1$

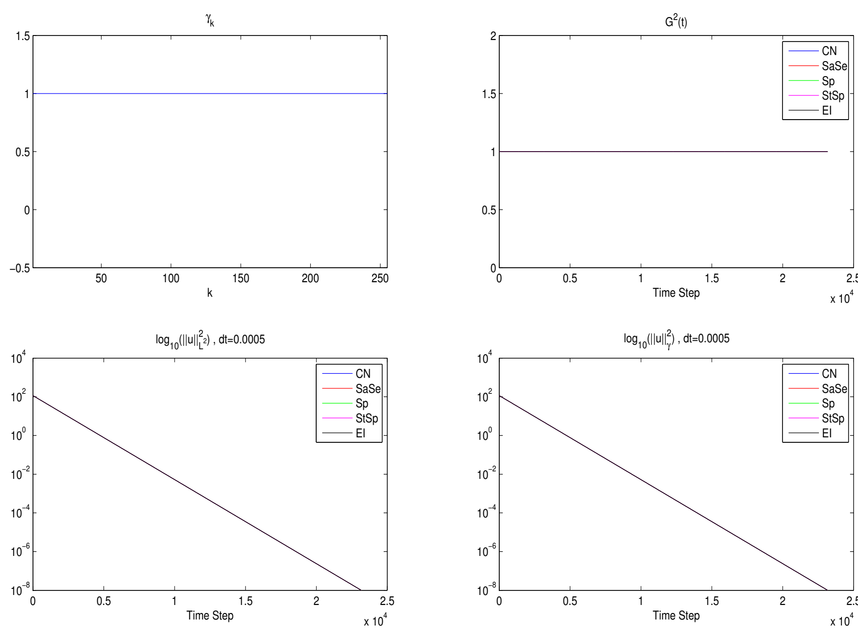


FIGURE 4. $u_0(x) = S1$, $\gamma_k = 1, \forall k$, $f(x) = 0$

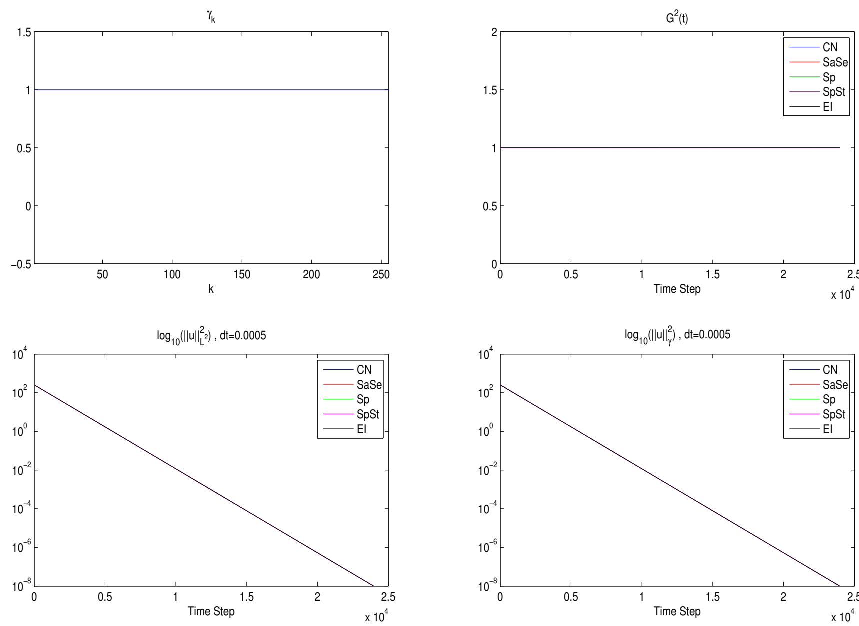


FIGURE 5. $u_0(x) = S3$, $\gamma_k = 1, \forall k$, $f(x) = 0$

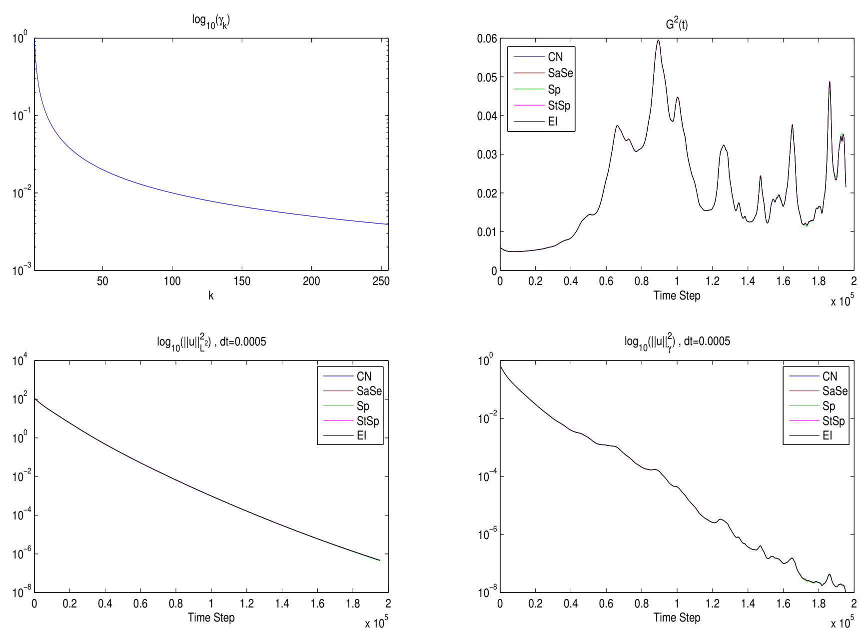


FIGURE 6. $u_0(x) = S1$,$\gamma_k = \frac{1}{(1+|k|)}$, $f(x) = 0$

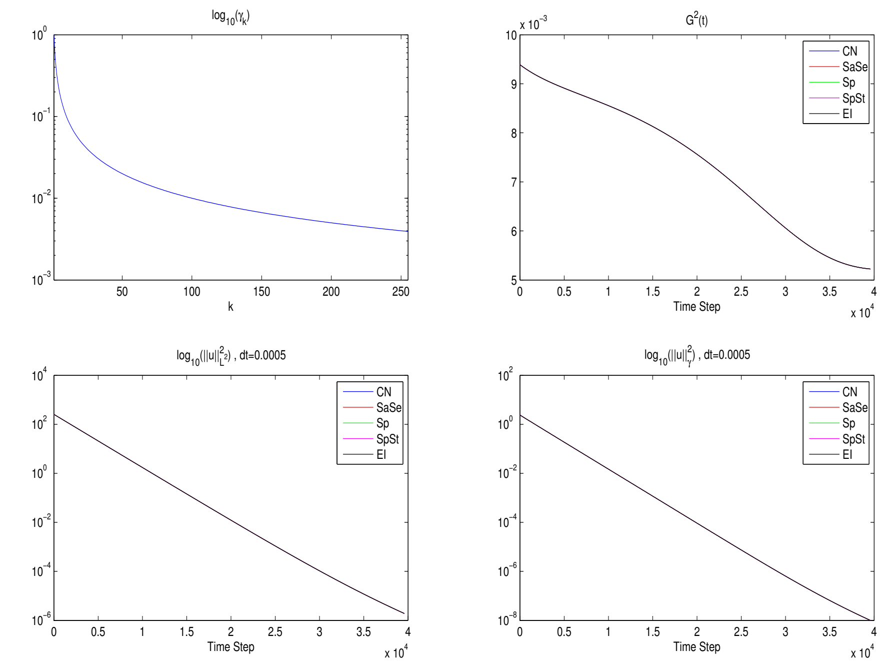


FIGURE 7. $u_0(x) = S3, \gamma_k = \frac{1}{(1+|k|)}$, $f(x) = 0$

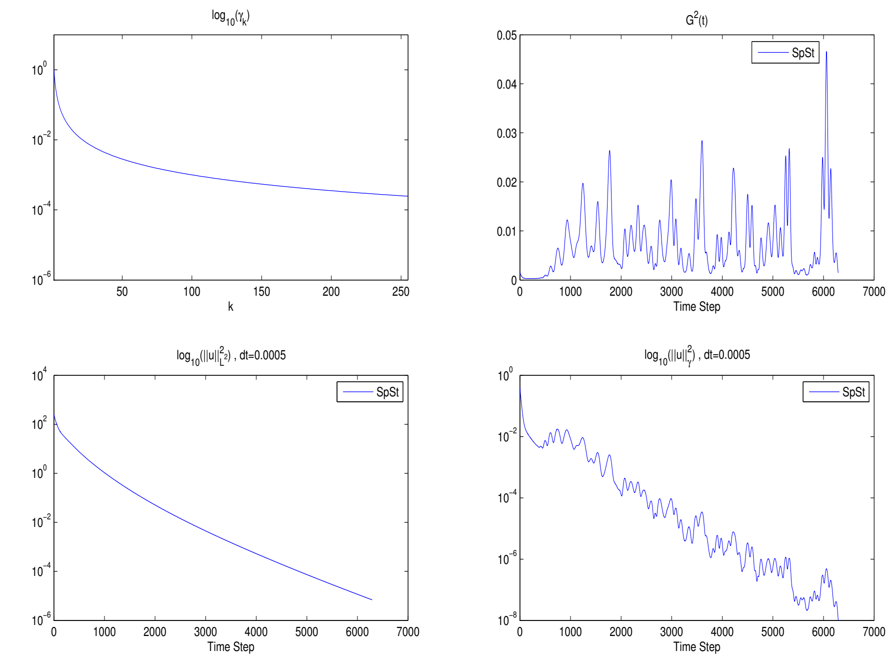


FIGURE 8. $u_0(x) = S1$,$\gamma_k = \frac{1}{(1+|k|)^2}$, $f(x) = 0$

4.3. **Forced KdV.** We now consider the forced and damped equation. We first present numerical results on the computation of steady states, time-periodic solutions. Then we give numerical evidences of the regularization properties by displaying the time evolution of the Sobolev regularity of the discrete solution.

4.3.1. *Steady state Solutions.* First of all we underline that it is difficult to obtain bounds on γ that guarantee the existence of steady state, as proposed in [3] for the case $\gamma = cste$. Indeed, assume that the equation possesses a stationary solution v, we set $w = u - v$ and we find that

$$w_t + \mathcal{L}_\gamma(w) + w_{xxx} + uu_x - vv_x = 0.$$

Hence, by taking the scalar product in L^2 with w

$$\frac{1}{2}\frac{d|w|_2^2}{dt} + |w|_\gamma^2 + \frac{1}{2}\int_0^L v_x w^2 dx = 0.$$

When $\gamma_k = cste$ or $\gamma_k \geq c > 0$, the equation is (at least) weakly damped and Rosa and Cabral [3] have pointed out non trivial dynamics (multiple stationary solutions, periodic solutions). In such a case,

$$|w|_\gamma^2 \geq c|w|_{L^2}^2,$$

hence, if there exists a strictly positive constant κ such that

$$c + \frac{v_x}{2} \geq \kappa > 0, a.e. \text{ in } [0, L],$$

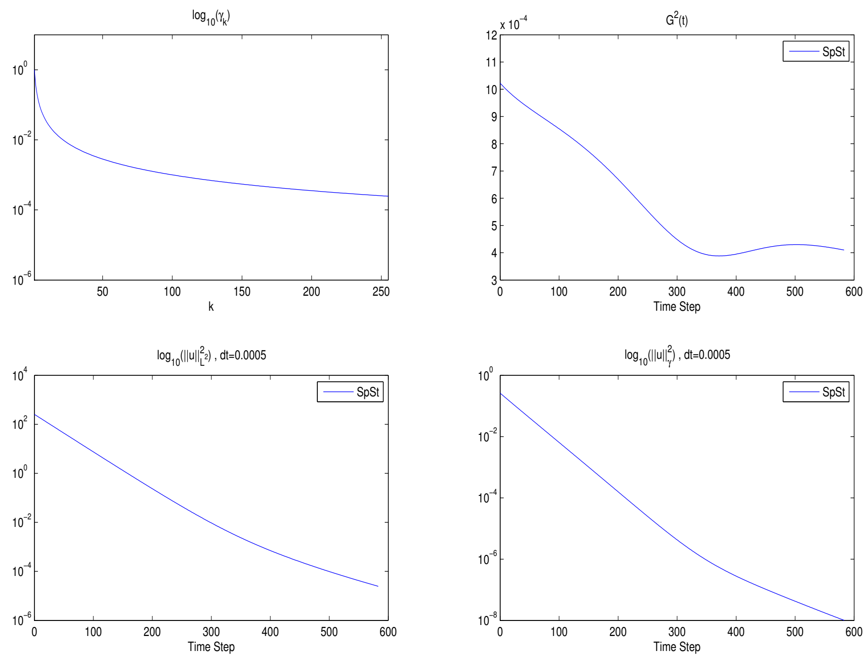


FIGURE 9. $u_0(x) = S3$, $\gamma_k = \frac{1}{(1+|k|)^2}$, $f(x) = 0$

then

$$|w|_{L^2} \leq e^{-\kappa t}|w_0|_{L^2},$$

so v is the only one steady state and it is asymptotically stable.

When γ_k converge towards 0 it is not possible to proceed as above because, $|.|_\gamma$ and $|.|$ are not equivalent norms. However, when f is constant, $u = \dfrac{f}{\gamma_0}$ is a steady state of the equation and the only one steady solution with mean equal to zero is $u = 0$.

The nontrivial long time dynamics occurs when the steady state u^* is not unique. A uniqueness condition is

$$|w|_\gamma^2 + \frac{1}{2}\int_0^L v_x w^2 dx \geq 0.$$

We have

$$\widehat{v_x w^2} = \widehat{v_x} * \widehat{w} * \widehat{w}.$$

Setting $\theta = v_x w^2$, we get

$$\int \theta dx = \frac{L}{2}\widehat{\theta}_0,$$

so

$$\widehat{\theta}_0 = \sum_{k\in\mathbb{Z}} \frac{-2i\pi k}{L}\widehat{v}_{-k}\left(\sum_{n\in\mathbb{Z}} \widehat{w}_{k-n}\widehat{w}_n\right).$$

Finally the decreasing condition reads as

$$\sum_{k\in\mathbb{Z}} |\widehat{w}_k|^2 + \sum_{k\in\mathbb{Z}} ik\pi\widehat{v}_k \left(\sum_{n\in\mathbb{Z}} \widehat{w}_{-k-n}\widehat{w}_n\right) \geq 0.$$

When bounding from above the second term with the Cauchy-Schwarz inequality, we obtain

$$\sum_{k\in\mathbb{Z}} |\widehat{w}_k|^2 - \sum_{k\in\mathbb{Z}} |k\pi\widehat{v}_k| \left(\sum_{n\in\mathbb{Z}} \widehat{w}_n^2\right) \geq 0.$$

Noting that

$$|v_x| = |\sum_{k\in\mathbb{Z}} \frac{2ik\pi}{L}\widehat{v}_k e^{\frac{2ik\pi x}{L}}| \leq \sum_{k\in\mathbb{Z}} \frac{2k\pi}{L}|\widehat{v}_k|,$$

we see that the decreasing condition can be satisfied only when γ_k are large enough, say $\gamma_k > \bar{\gamma} > 0$ for a suitable $\bar{\gamma}$, see also [3].

We represent in figure 10 the different steady states obtained for different sequences γ_k, starting with $u_0 = S3$ as initial data and taking $f = \sin(2\pi x/L)$. We

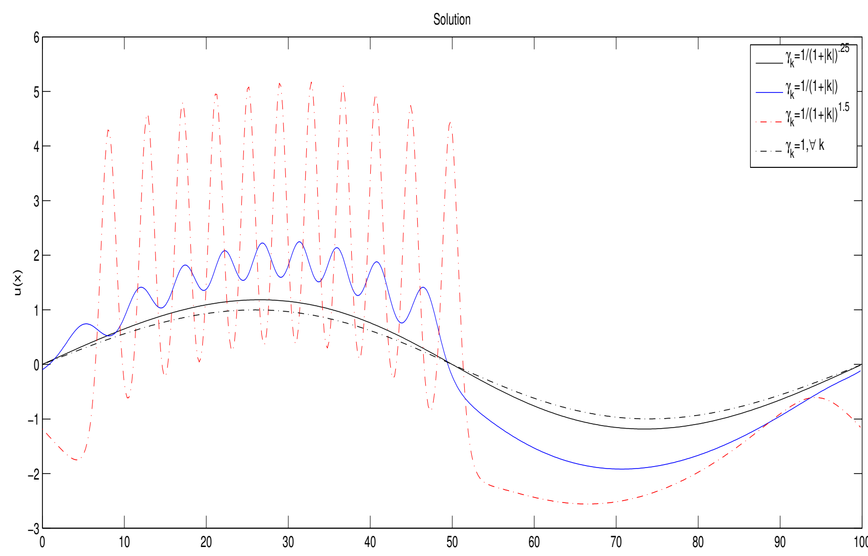


FIGURE 10. Computed steady states with different γ_k, $u_0(x) = S3$, $f(x) = \sin(2\pi x/L)$

display $\left\|\dfrac{du}{dt}\right\|_{L^2}$ as a way to measure the convergence to the steady sate. We observe in Figure 11, when $\gamma_k = 1$ (left) and $\gamma_k = \frac{1}{(1+|k|)^{.5}}$ (right) that the rate of convergence to the steady state depends on the initial data.

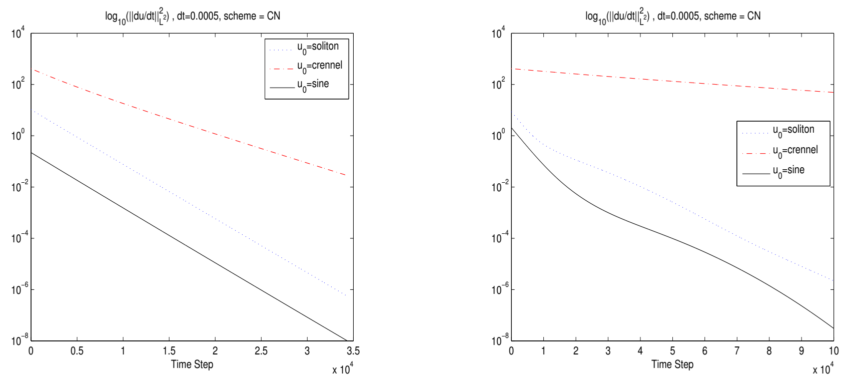


FIGURE 11. Computed steady states for $\gamma_k = 1, \forall k$ (left) and for $\gamma_k = \frac{1}{(1+|k|)^{.5}}$ (right), $f(x) = \sin(2\pi x/L)$, various $u_0(x)$: evolution of $\|\frac{du}{dt}\|^2_{L^2}$.

4.3.2. *Time periodic solutions.* The computation of time periodic solutions needs to start from an appropriate initial datum. We restrict to the case $L = 2\pi$ and we chose $u0(x) = S4 = 50\chi_{x>\pi}\sin(4x)$: this is the initial data to compute time-periodic solutions and is inspired by that use by Rosa and Cabral [3]. A way to observe numerically the time periodicity is to display phase portrait for different pair of frequencies and to point out close trajectories. The new result here, reported in Figures 12 and 13 is that we have still time periodic solutions, even $\lim_{|k|\to+\infty} \gamma_k = 0$: this result traduces a non trivial dynamics for large times.

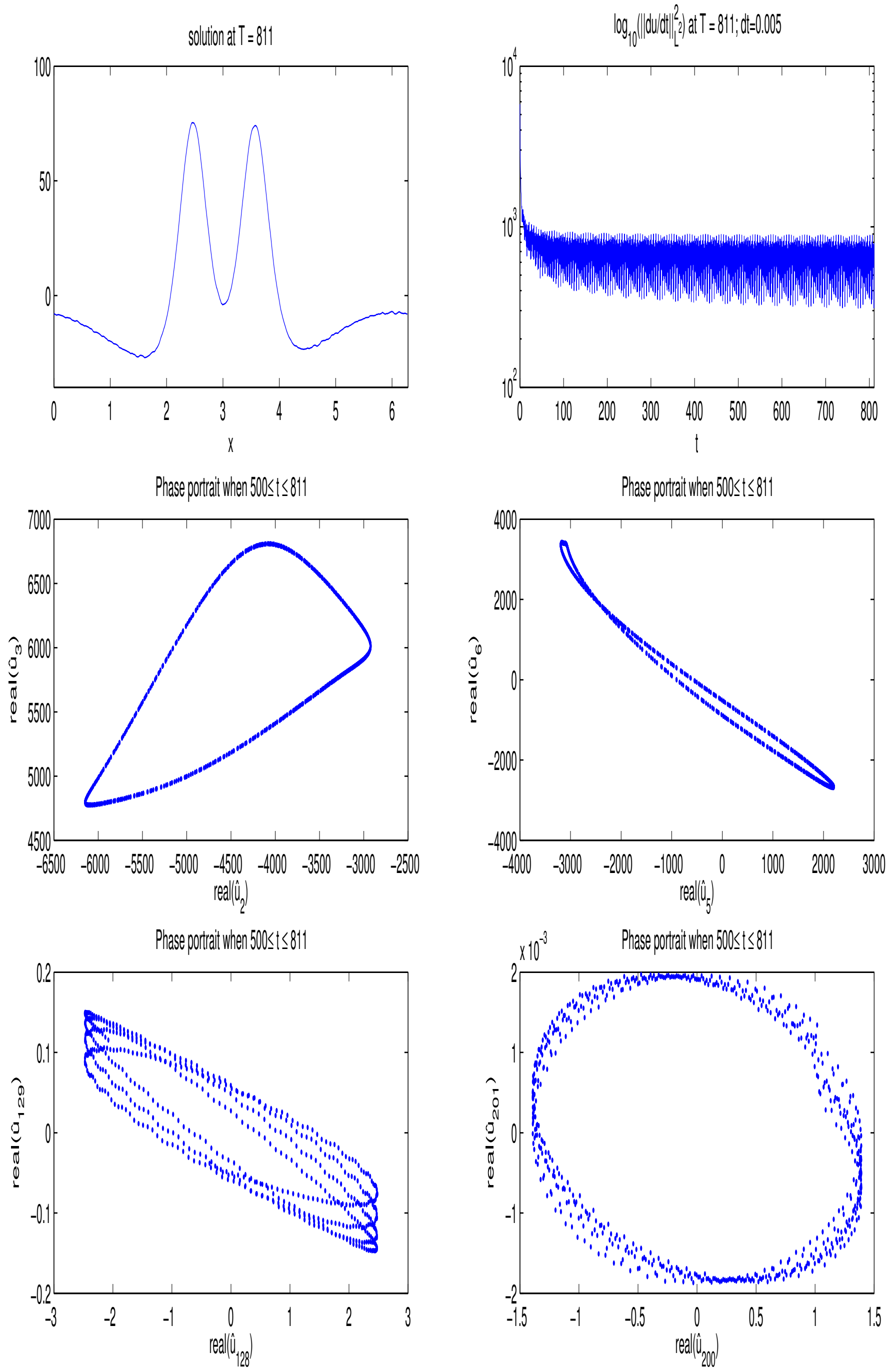


FIGURE 12. $u_0(x) = S4$,$\gamma_k = 2.7, \forall k,\ f(x) = \sin(2\pi x/L)$

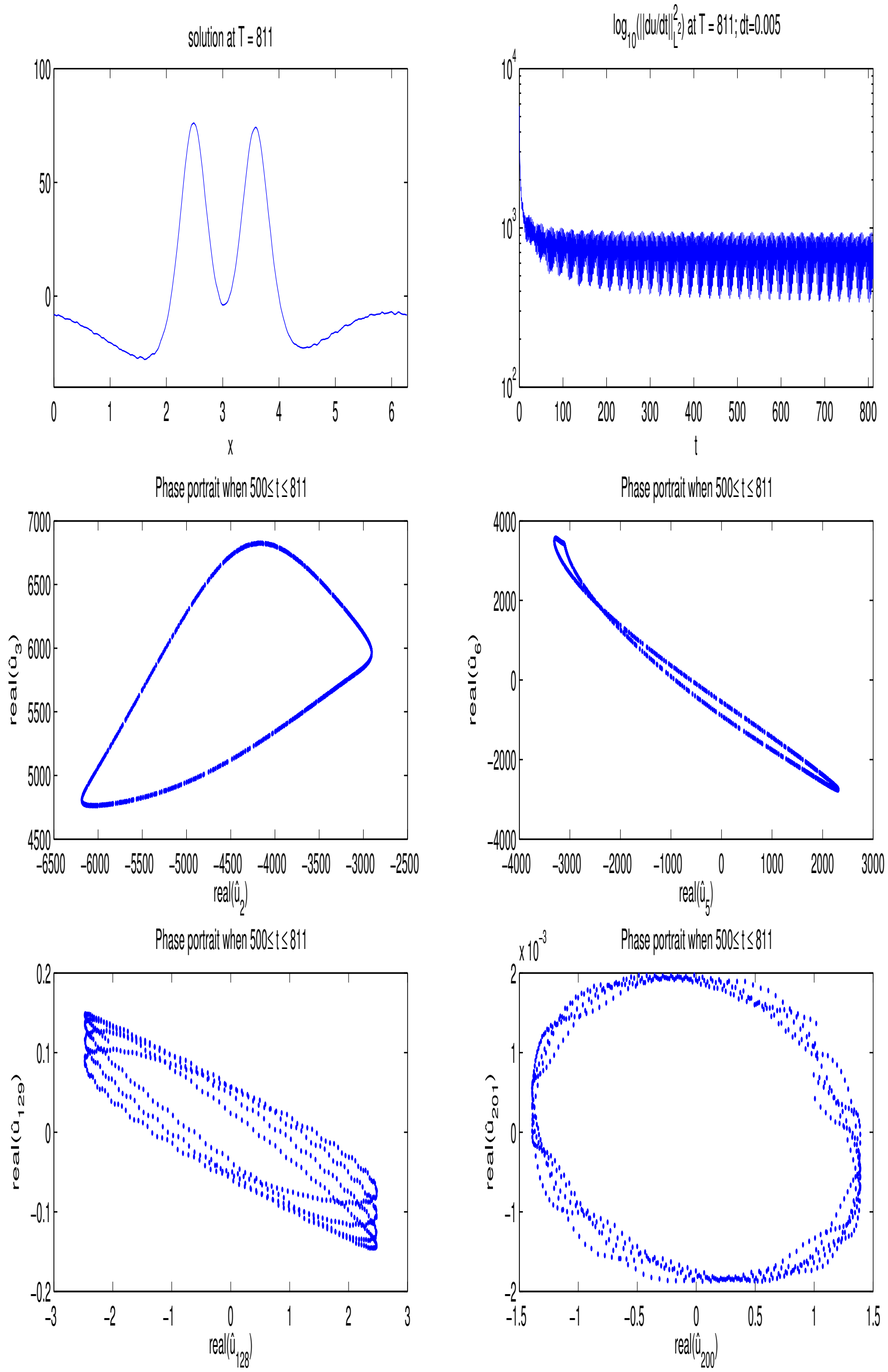


FIGURE 13. $u_0(x) = S4$,$\gamma_k = \dfrac{3.8}{(1+|k|)^{0.3}}$, $f(x) = \sin(2\pi x/L)$

4.3.3. *Long time behavior : measure of the numerical Sobolev regularity.* As said in the introduction, one of the aims of the present work is to observe an asymptotic regularization of the solutions. This have been established mathematically for the so-called weak damping ($\gamma_k = c$) [10, 11], but it remains to be pointed out for $\gamma_k \to 0$. To this end we capture the discrete smoothing by analyzing the convergence of the truncated Fourier expansion of the solution. More precisely if, for a fixed time t, we expand $u(x,.)$ as

$$u(x,.) = \sum_{k=1}^{+\infty} \widehat{u}_k w_k(x), \tag{32}$$

where $w_{2k}(x) = \sin(2k\pi x)$, $k \geq 1$, $w_{2k+1}(x) = \cos(2k\pi x)$, $k \geq 0$.
The smoothness of the function $u(x)$ shows on the decay of the high frequency modes. As in [1], we follow [13] starting from the fact that a function u that is in L^2 belongs to H^s, $s > 0$ iff

$$\sum_{N=1}^{+\infty} N^{2s-1} ||u - u_N||^2_{L^2} < +\infty, \tag{33}$$

where

$$u_N = \sum_{|k|>N} \widehat{u}(k) e_k(x). \tag{34}$$

Of course, in practice, this formula will be applied with a finite number of Fourier modes. If we use two levels of approximations, the fine one u_N which will play the role of u in (33) and the coarse one $u_{N/2}$ which will play the role of u_N in (33), we have to compute s such that

$$\sum_{k=1}^{N/2} k^{2s-1} \left(\sum_{\ell=k}^{N} |\widehat{u}_k|^2 \right) < +\infty.$$

The numerical Sobolev exponent is computed by considering the queue of the spectral energy of the solution. More precisely, we look to a linear behavior of the high frequencies Fourier coefficients as

$$\sum_{\ell=k}^{N} |\widehat{u}_\ell|^2 \simeq \frac{C}{k^{2s}}, \text{ for } k \gg 1,$$

or equivalently

$$v_k = \ln \sum_{\ell=k}^{N} |\widehat{u}_\ell|^2 \simeq \ln(C) - 2s \ln k, \text{ for } k \gg 1.$$

It suffices then to compute s by a linear regression (least square fitting). In practice, we will select a few number (m) of $\widehat{u}_k$. Hence s (and $\kappa = \ln(C)$) are computed as minimizers of

$$\sum_{k=N-m}^{N} (v_k - (\kappa - 2s\ln(k))^2.$$

In Figures 14, 15, 16, 17 we point out numerically regularization effects even when $\lim_{k\to+\infty} \gamma_k = 0$: the numerical Sobolev regularity increases with the time. This property is crucial for developing multilevel methods based on the splitting of the solution into low and high mode components, see [4].

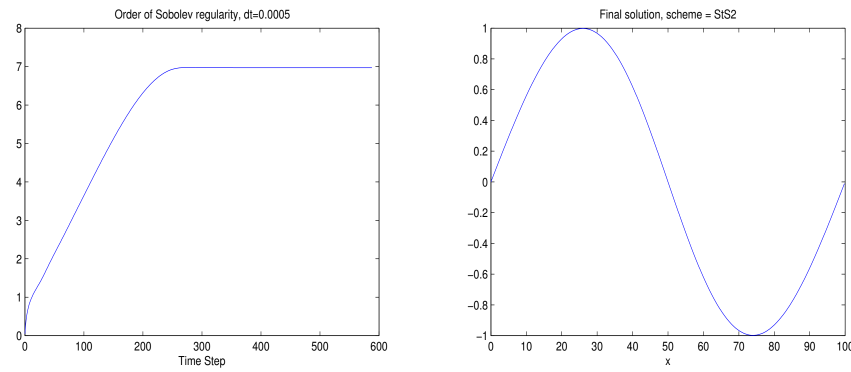


FIGURE 14. $\widehat{u}_k(0) = 1/(1+|k|)$ if k is even, 0 if k is odd, $\gamma_k = 1, \forall k, f(x) = \sin(2\pi x/L)$

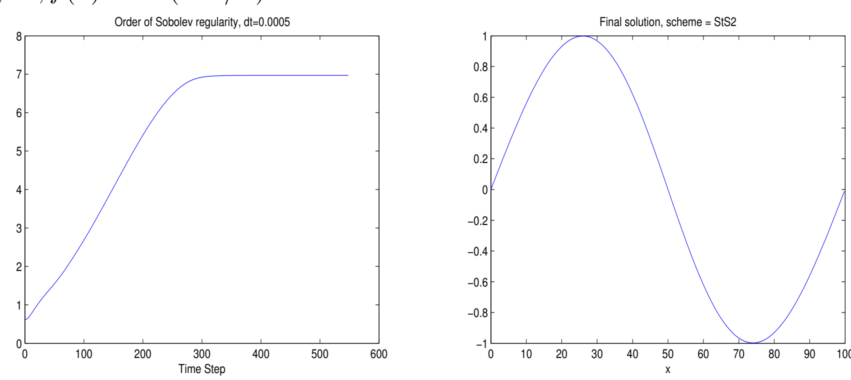


FIGURE 15. $u_0(x) = S2$,$\gamma_k = 1, \forall k$ $f(x) = \sin(2\pi x/L)$

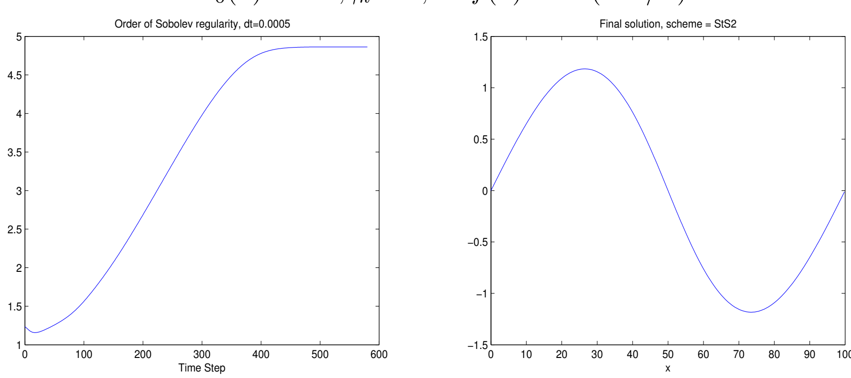
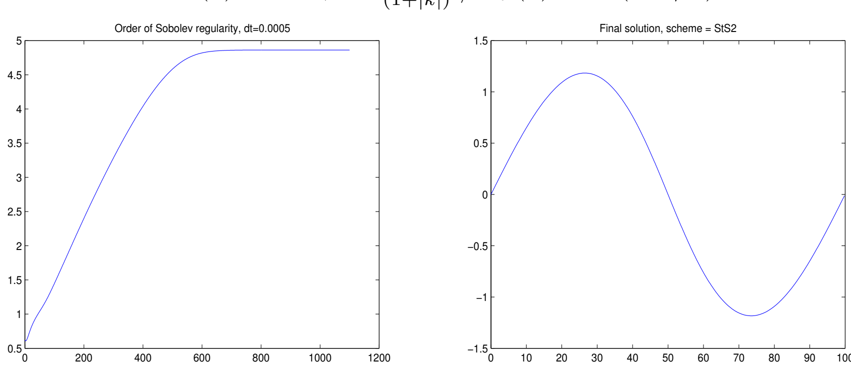


FIGURE 16. $u_0(x) = S1$,$\gamma_k = \frac{1}{(1+|k|)^{1/4}}$, $f(x) = \sin(2\pi x/L)$



5. **Concluding remarks and perspectives.** The family of weak damped KdV equation we have presented here allows to consider damping weaker than those studied in [8, 9, 10, 11] in the sense that, when $\gamma_k \to 0$ as $|k| \to +\infty$,

$$\exists c > 0 \text{ s.t. } |u|_\gamma \leq c|u|_{L^2} \forall u \in L^2 \text{ and } \nexists d > 0 \text{ s.t. } d|u|_\gamma \geq |u|_{L^2} \forall u \in L^2,$$

for which an asymptotic regularization property was proved. The numerical illustrations show that a damping is still present in energy norms ($|.|_{L^2}$ and $|.|_\gamma$), the main problem being that we do not have a uniform control on the rate of damping. Anyway, and as in the weak damping case considered by Cabral and Rosa ([3]) (see also [2]), we pointed out by numerical evidence, the existence of steady states and of time-periodic solutions, which means a nontrivial long time dynamics. As possible perspectives of the present work, we propose:

1. The rough mathematical analysis of the long time behavior of (5) still remain to be done; it presents technical difficulties due to the non embedding of H_γ in L^2 when $\lim\limits_{k \to +\infty} \gamma_k = 0$, it is then helpful to produce first numerical evidences.
2. We can consider such damped KdV equations when the boundary conditions are not periodic. It suffices to expand the solution in a proper (Hilbert) orthogonal polynomial basis p_k with respect to a weighted L^2 scalar product $(.,.)_\omega$. We have then
$$u(x,t) = \sum_{k=0}^{\infty} \widehat{u}_k p_k(x),$$
where $\widehat{u}_k = \dfrac{(u, p_k)_\omega}{(p_k, p_k)_\omega}$ and we define the damping linear operator $\mathcal{L}_\gamma$ as
$$\mathcal{L}_\gamma(u) = \sum_{k=0}^{\infty} \gamma_k \widehat{u}_k p_k(x).$$
More widely, the study of the long time behavior of equations as
$$\frac{\partial u}{\partial t} + \mathcal{L}_\gamma(u) + F(u) = 0,$$
with $Re(F(u), u)_\omega = 0$ and $\mathcal{L}_\gamma$ defined as above can be addressed following the same approach. For example damped Nonlinear Schrodinger equations as presented in [1] can be considered.
3. In a numerical point of view, the damping when treated implicitly in a numerical marching scheme, has a stabilization property. The Fourier expression of $\mathcal{L}_\gamma$ allows to study in a accurate way the effect of the damping by band of frequency: this can be a first step before building filtering operators for stabilizing numerical scheme without deteriorating the consistency, as done e.g. in [5] for nonlinear parabolic equations.

We hope to develop these topics in a near future.

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